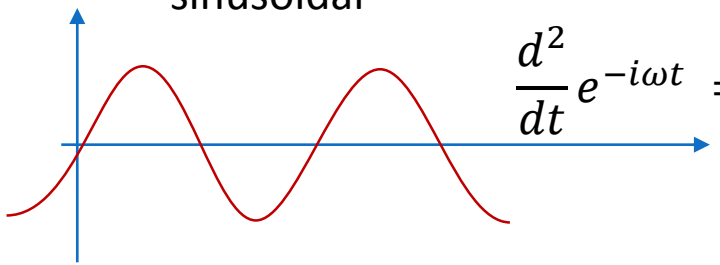
Constructing and Verifying Cyber Physical Systems

Control and Feedback

Marcus Völp

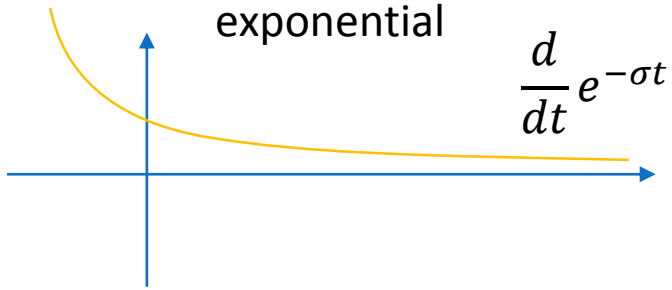
Laplace Transform

sinusoidal

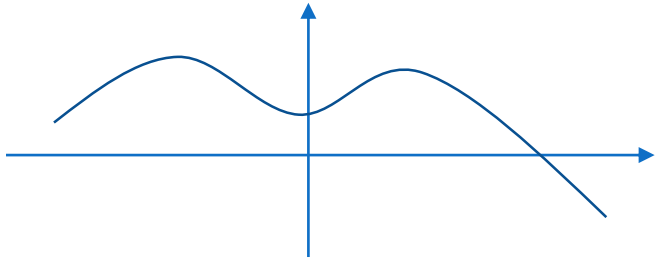


$$\frac{d^2}{dt} e^{-i\omega t} = -\omega^2 \, e^{-i\omega t}$$

exponential



$$\frac{d}{dt} e^{-\sigma t} = -\sigma \, e^{-\sigma t}$$



… σ=-1

premultiply exponential σ=0

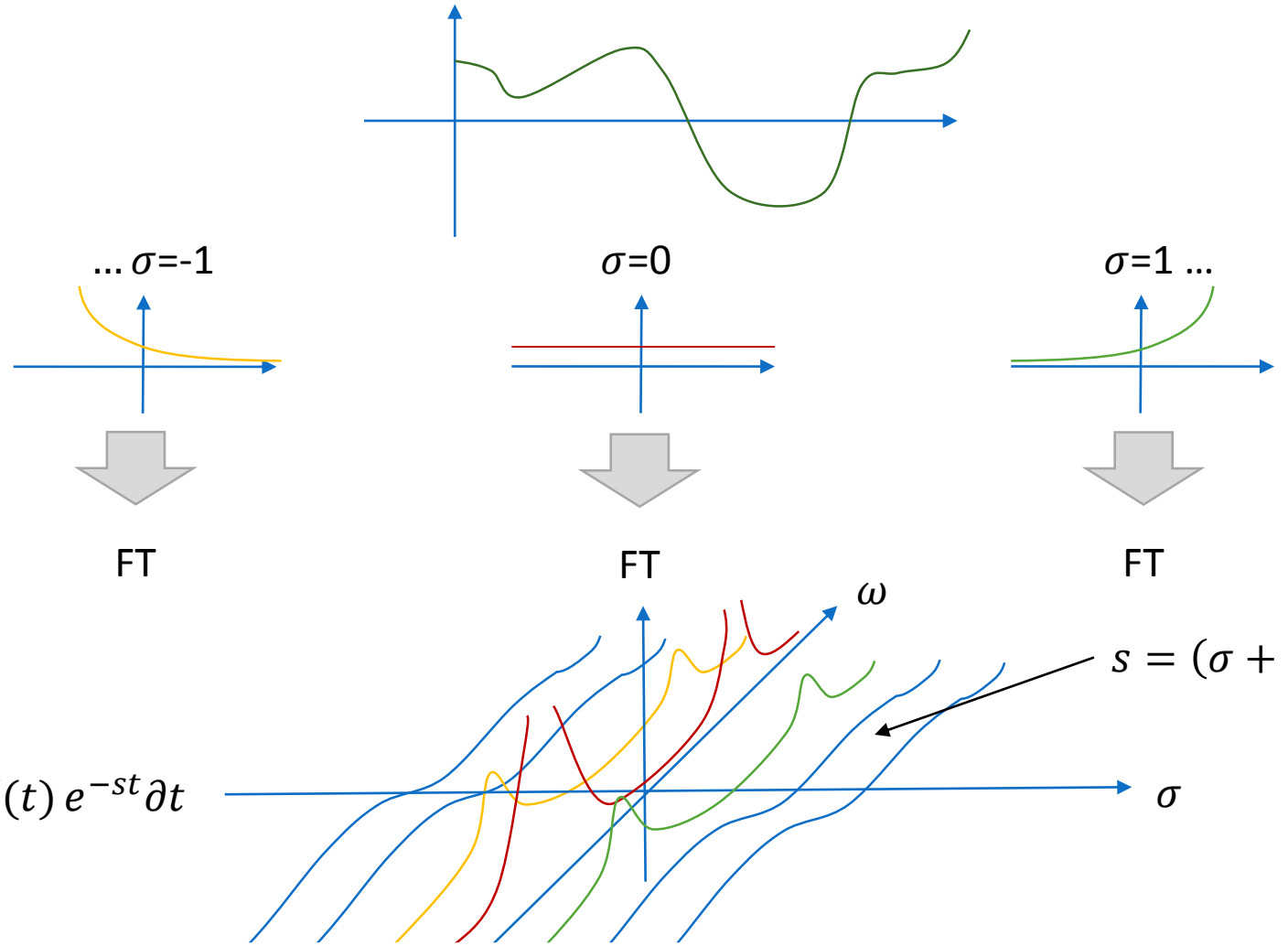
σ=1 …

FT

FT

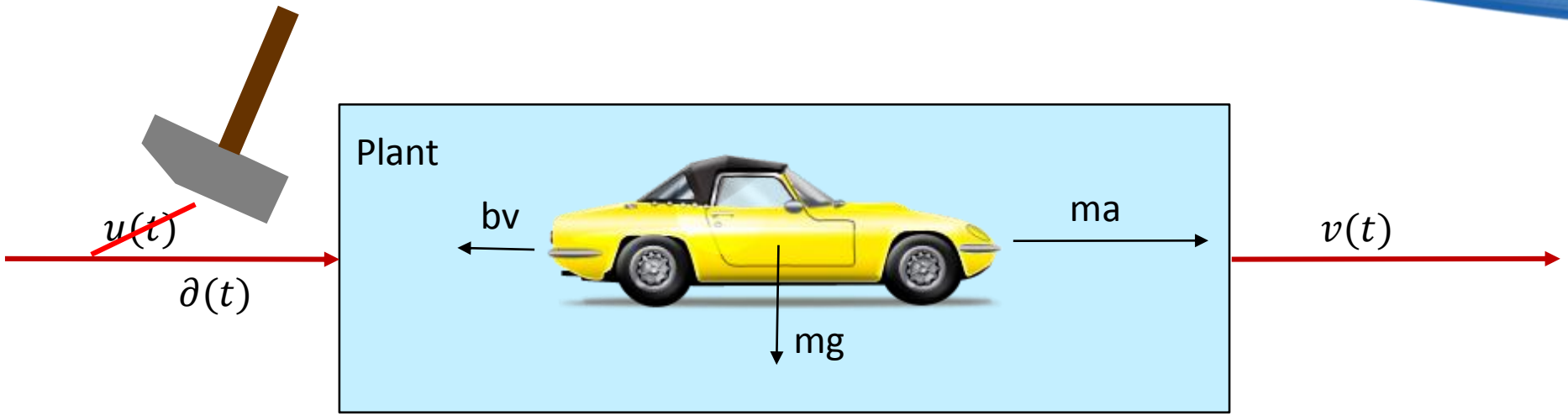
FT

Laplace Transform



$$\mathcal{L}\_(f)(s) = \int_{0^-}^{\infty} f(t)\, e^{-st} \partial t$$

Laplace Transform



$$v(t) = v_\partial(t) * u(t)$$

$\mathcal{L}\_(v_\partial)(s)$ $\mathcal{L}\_(u)(s)$

$$V(s) = \frac{\frac{1}{m}}{s + \frac{b}{m}} \cdot U(s)$$

$$v(t) = \mathcal{L}^{-1}\_(V)(t)$$

Exercise

$$\mathcal{L}\_(f)(s) = \int_{0^-}^{\infty} f(t)\, e^{-st} \partial t$$

superposition

$$\mathcal{L}_{(af(t)+bg(t))(s)} = \int_{0^-}^{\infty} \big(af(t) + bg(t)\big)\, e^{-st} \partial t = a \int_{0^-}^{\infty} f(t)\ e^{-st} \partial t + b \int_{0^-}^{\infty} g(t)\ e^{-st} \partial t = a\mathcal{L}_{(f)(s)} + b\mathcal{L}_{(g)(s)}$$

time delay

$$\mathcal{L}_{(f(t-\lambda))(s)} = \int_{0^-}^{\infty} f(t-\lambda)\, e^{-st} \partial t \overset{t=x+\lambda}{\cong} \int_{0^-}^{\infty} f(x)\, e^{-sx-s\lambda} \partial x = e^{-s\lambda} \int_{0^-}^{\infty} f(x)\ e^{-sx} \partial x = e^{-s\lambda} \mathcal{L}_{(f)(s)}$$

frequency shift

$$\mathcal{L}_{(e^{-at} f(t))(s)} = \int_{0^-}^{\infty} e^{-at} f(t)\, e^{-st} \partial t = \int_{0^-}^{\infty} f(t)\ e^{-(s+a)t} \partial t = \mathcal{L}_{(f)(s+a)}$$

convolution

$$\mathcal{L}_{(f(t)*g(t)\,)(s)} = \int_{0^-}^{\infty} \int_{-0}^{\infty} f(\tau) g(t-\tau)\, \partial\tau\, e^{-st} \partial t = \int_{0^-}^{\infty} f(\tau) \left(\int_{-0}^{\infty} g(\sigma)\, e^{-s\sigma} \partial\sigma \right) e^{-s\tau} \partial\tau = \mathcal{L}_{(f)(s)} \mathcal{L}_{(g)(s)}$$

Overview

Introduction	Math
Mathematical Foundations (Differential Equations and Laplace Transformation)	
Control and Feedback	Physics
Transfer Functions and State Space Models	
PID Control	Feedback Control
Stability and Root Locust Method	
Mixed-Criticality Scheduling and Real-Time Operating Systems (RTOS)	RTOS
Program Verification	Verification
Differential Dynamic Logic and KeYmaera X	
Differential Invariants	

Overview

Control

Regulating Control **Tracking Control**

Open Loop Control **Closed Loop Control**

Block Diagrams / Simulink

↓

Control Equations

↓

Robustness

↓

Stability / Final Value Theorem

Noise

↓

Observability

↓

Sensor Fusion

G. Franklin, J.D. Powell, A. Emami-Naeini
"Feedback Control of Dynamic Systems"
7th ed. Pearson

K. Åström, R. Murray
"Feedback Systems –
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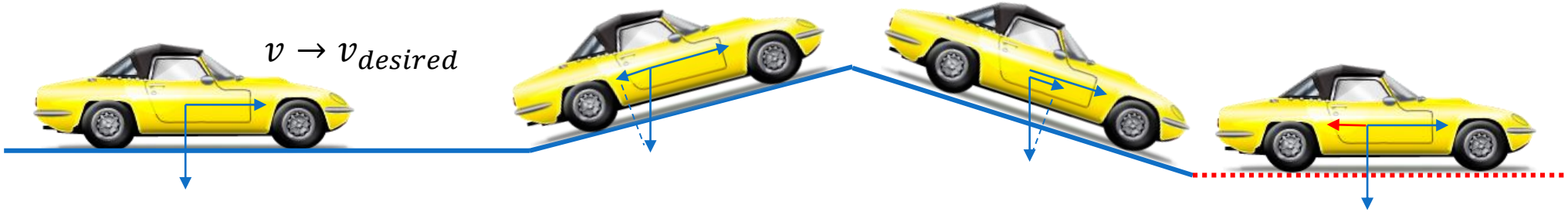
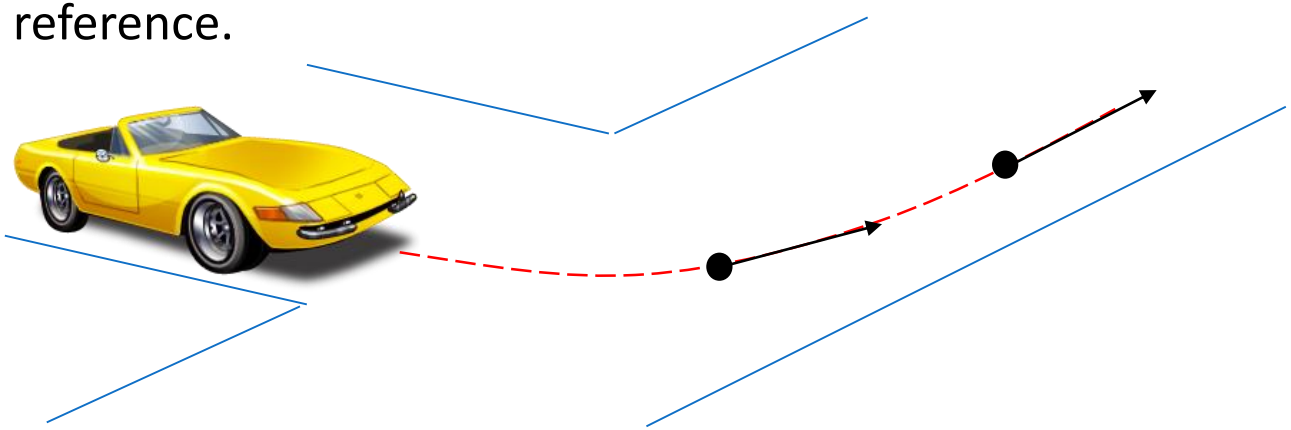
Princeton University Press

Control: adhere to reference value

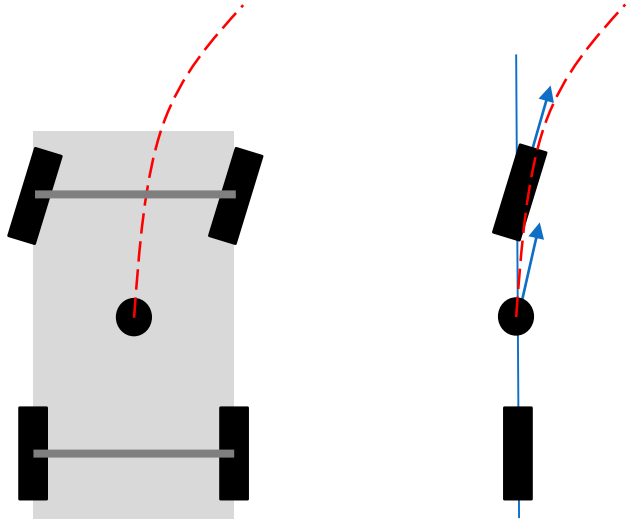
- Regulating Control (Regulator)
  Maintain desired output despite external disturbances.



- Tracking Control (Servo)
  Follow reference.

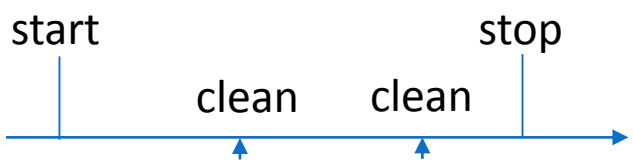


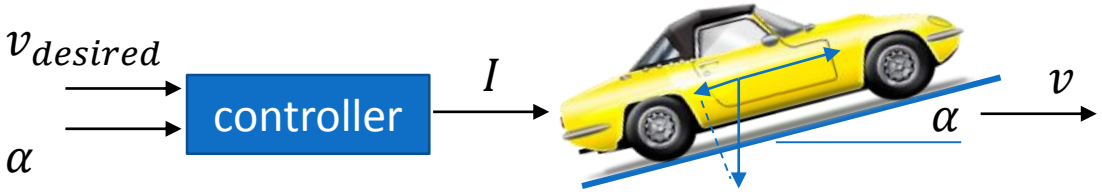
Bicycle model



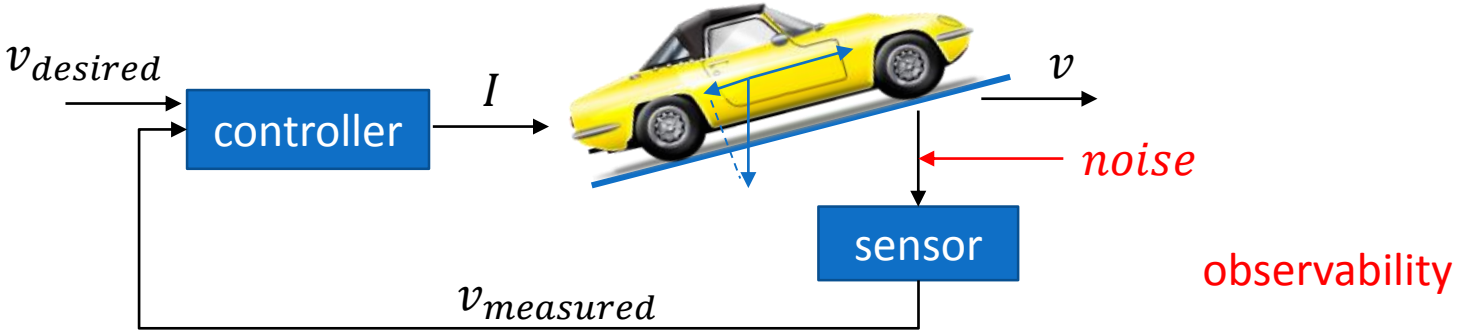
Feedback

- Open Loop Control / Feedforward Control
  no measurement of output / no correction to make output conform to reference signal



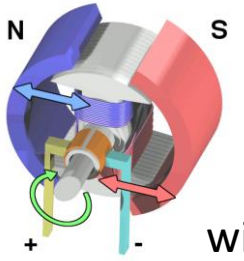


- Closed Loop Control / Feedback Control
  adjust input based on measured output



Modeling Concepts

- Heritage of Mechanics
  Describe the evolution of the system state with ordinary differential equations



cc-by-sa 3 (Wapcaplet)

winding resistance

motor's back EM force constant

$$U = L\frac{\partial I}{\partial t} + RI + k_b\theta'$$
$$M\theta'' = k_T I - v\theta' - \tau$$

torque

rotor's moment of inertia

torque constant

$\mathcal{L}$

$$\frac{T(s)}{U(s)} = \frac{\frac{k_T}{L}}{s + \frac{R}{L}}$$

tire

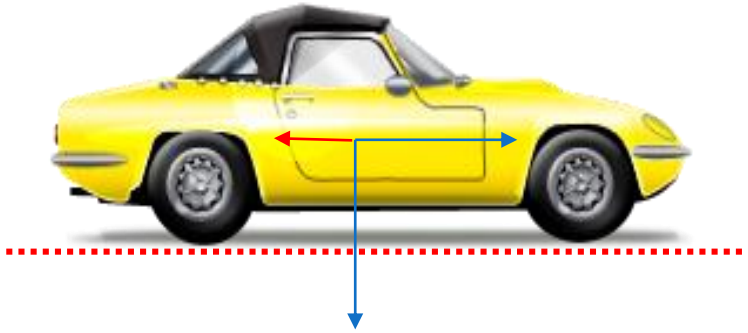

cc-by-sa 3 (M93)

$\tau = r \times F$

$*$

$\mathcal{L}$

$\cdot$



$$v' + \frac{b}{m}v = \frac{F_0}{m}\partial(t)$$

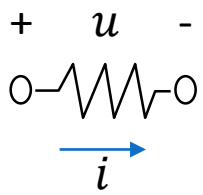
$\mathcal{L}$

$$\frac{V(s)}{U(s)} = \frac{\frac{1}{m}}{s + \frac{b}{m}}$$

Modeling Concepts

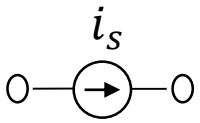
- Heritage of Electrical Engineering
  Describe the input/output behavior of connected components

Resistor



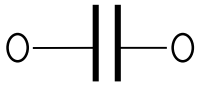
$u = Ri$

Current source



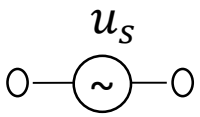
$i = i_s$

Capacitor



$i = C\frac{\partial u}{\partial t}$

Voltage source



$u = u_s$

Inductor

$u = L\frac{\partial i}{\partial t}$

Kirchhoff's current law (KCL):

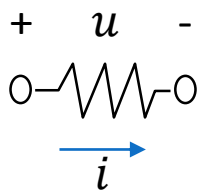
sum of currents leaving a junction or node equals sum of currents entering the node

Kirchhoff's voltage law (KVL):

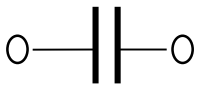
sum of voltages taken around a closed path in a circuit is zero

Modeling Concepts

- Heritage of Electrical Engineering
  Describe the input/output behavior of connected components

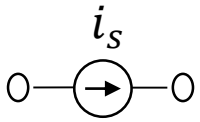


$$u = Ri$$

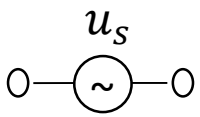


$$i = C\frac{\partial u}{\partial t}$$

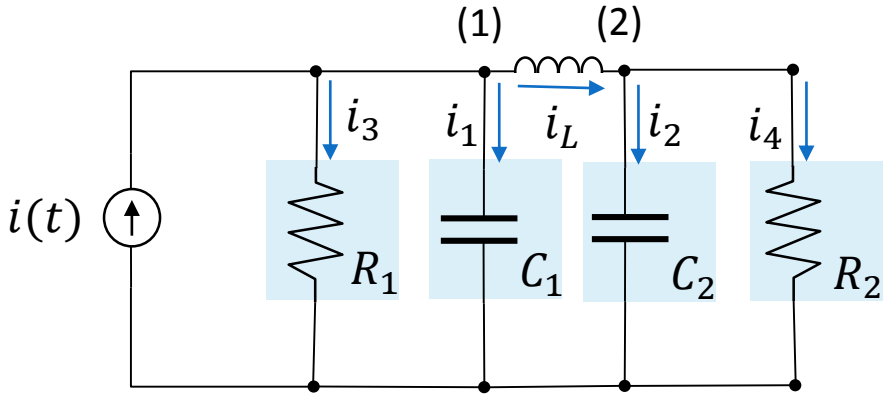
$$u = L\frac{\partial i}{\partial t}$$



$$i = i_s$$



$$u = u_s$$



$$i_1 = C_1\frac{\partial u_1}{\partial t}$$

$$i_2 = C_2\frac{\partial u_2}{\partial t}$$

$$i_3 = \frac{u_1}{R_1}$$

$$i_4 = \frac{u_2}{R_2}$$

$$u_1 - u_2 = L\frac{\partial i_L}{\partial t}$$

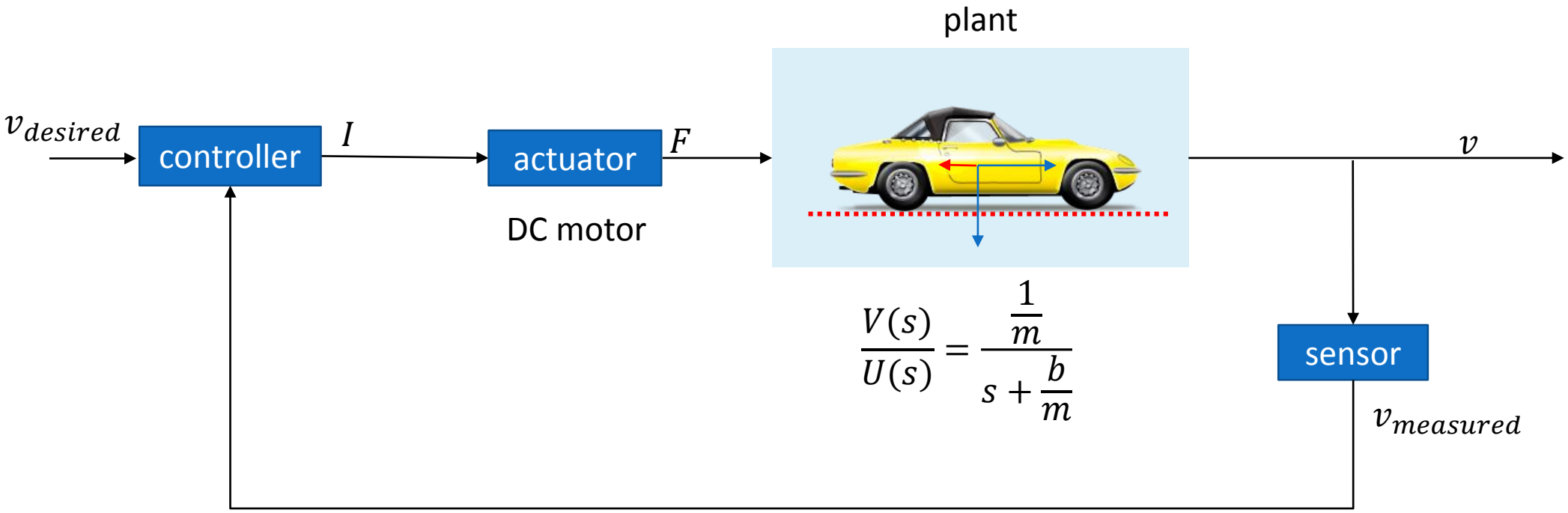
KCL =>

(1) $i(t) = i_3 + i_1 + i_L$

(2) $i_L = i_2 + i_4$

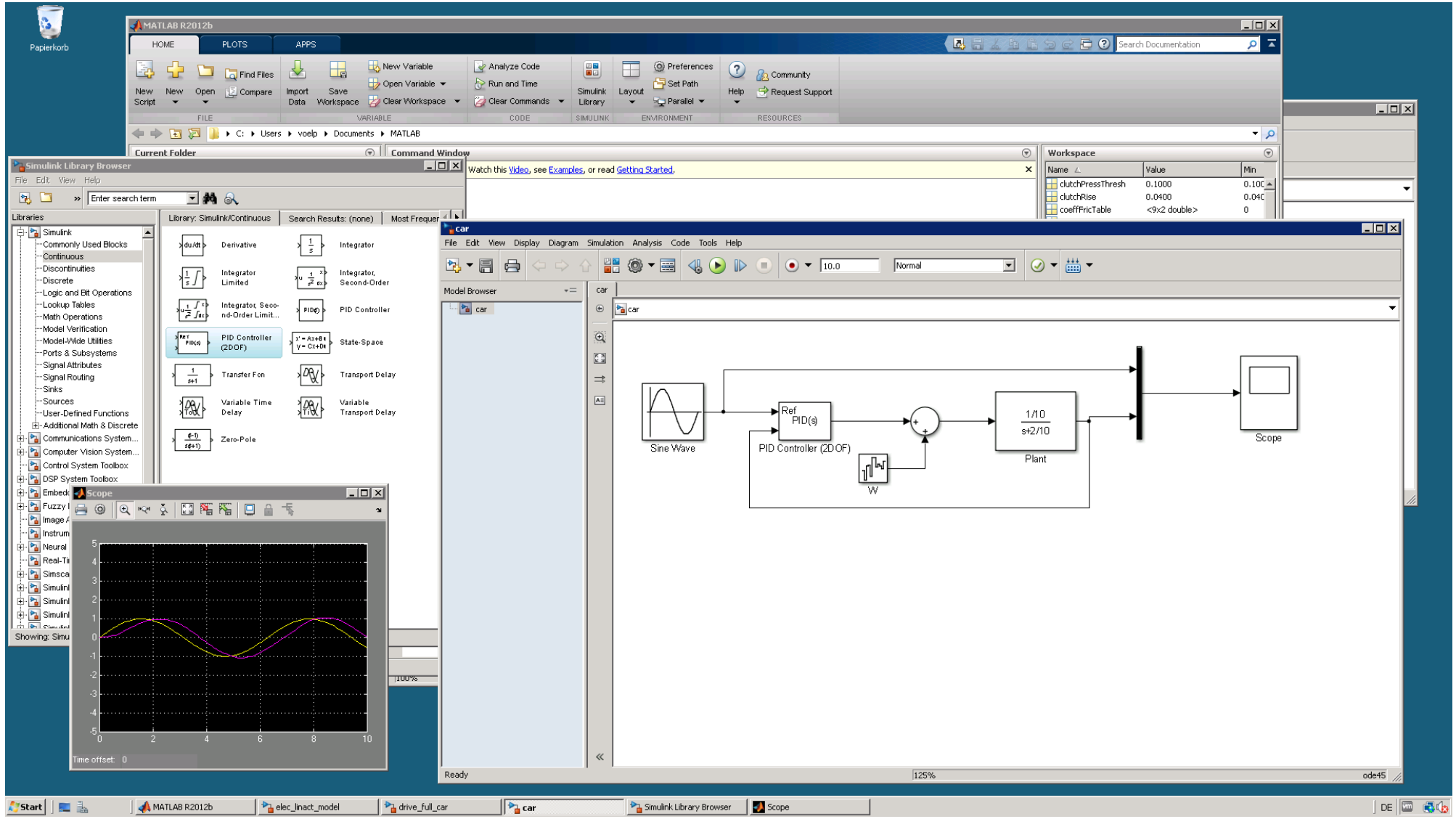
Modeling Concepts

- Heritage of Electrical Engineering
  Describe the input/output behavior of connected components



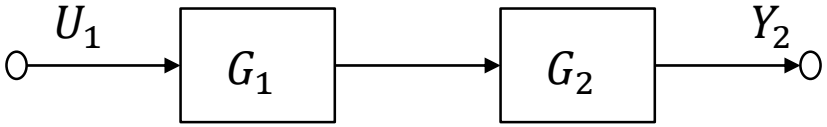
$$\frac{V(s)}{U(s)} = \frac{\frac{1}{m}}{s + \frac{b}{m}}$$

MatLab / Simulink

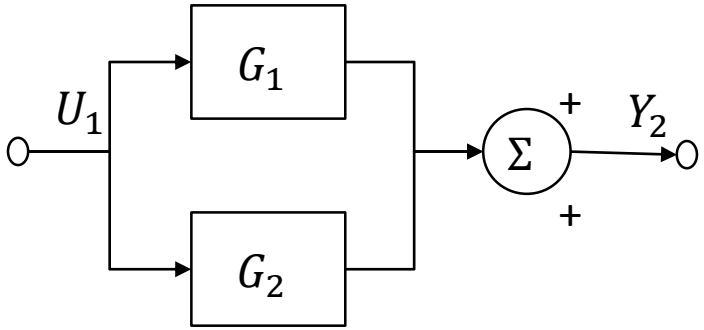


Block Diagrams

Open Loop Structures

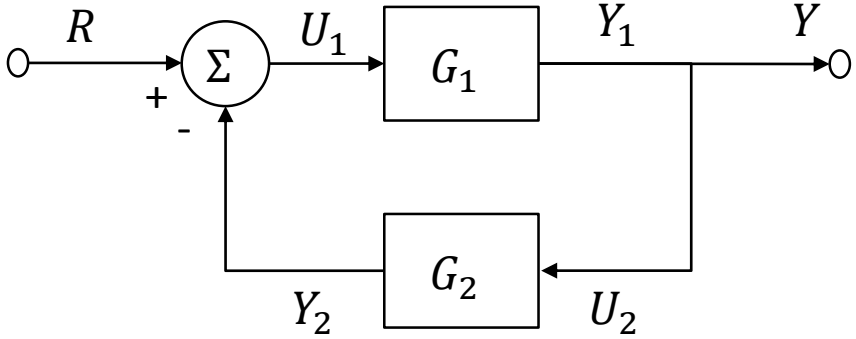


$$\frac{Y_2(s)}{U_1(s)} = G_2 G_1$$



$$\frac{Y_2(s)}{U_1(s)} = G_2 G_1$$

Closed Loop Structures



$$U_1(s) = R(s) - Y_2(s)$$

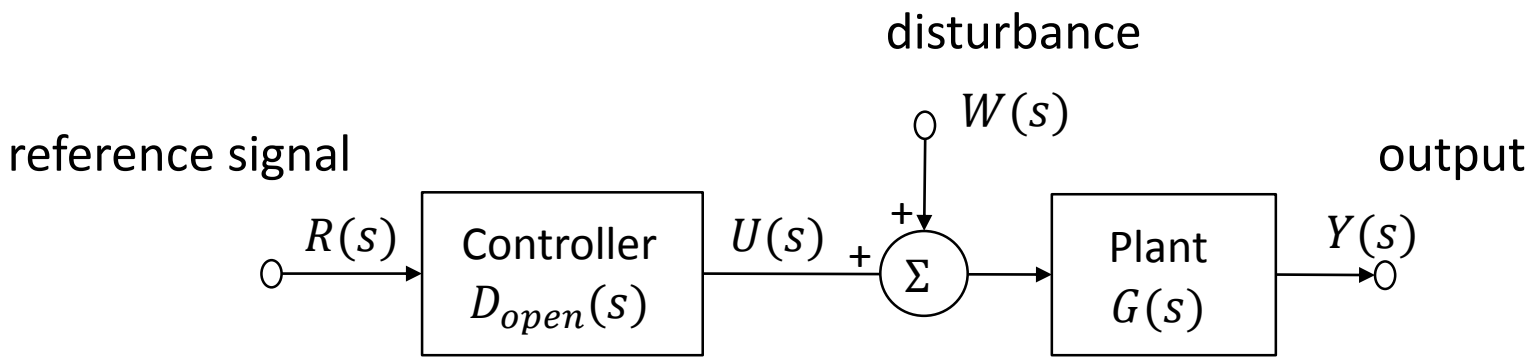
$$Y_2(s) = G_2(s) G_1(s) U_1(s)$$

$$Y_1(s) = G_1(s)\, U_1(s)$$

$$Y_1(s) = \frac{G_1(s)}{1 + G_1(s) G_2(s)} R(s)$$

Basic Equations of Control

Open Loop Control



$$Y = G\mathrm{DR} + \mathrm{GW}$$

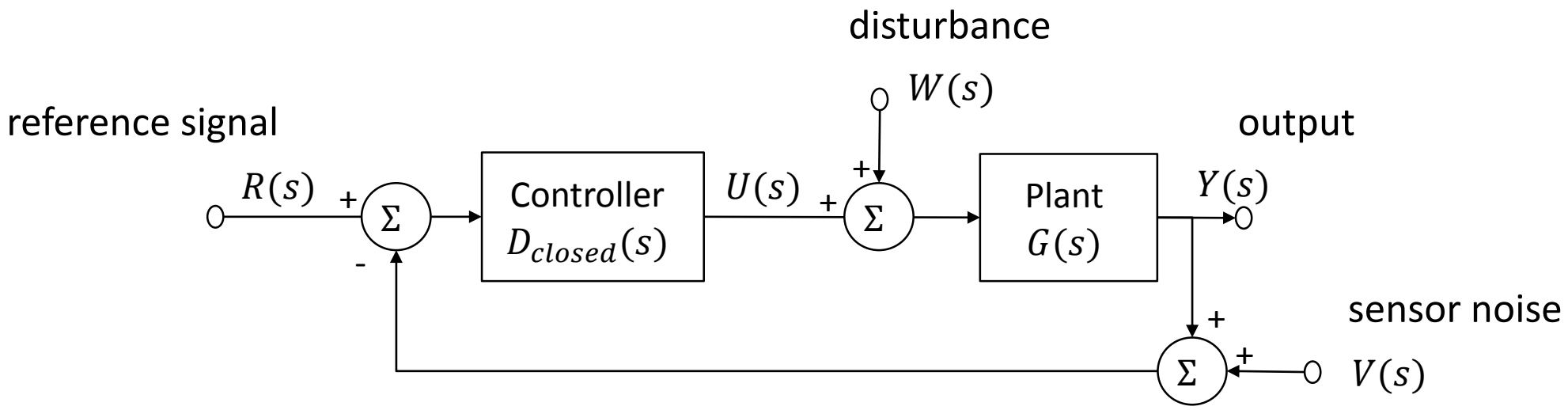
error

$$E = R - Y = [1 - GD]R - GW$$

- How good does the controller track the reference R?
- How much does it overshoot?
- Will the output eventually settle to the reference signal?
- How well does it compensate disturbances W to the plant?
- How sensitive is the controller to variations in the mathematical model?

Basic Equations of Control

Closed Loop Control



$$Y = T\mathrm{R} + \mathrm{GSW} - \mathrm{TV}$$

<u>error</u>

$$E = SR - GSW + TV$$

$$S = \frac{1}{1 + GD} \qquad T = GDS = \frac{GD}{1 + GD}$$

- How good does the controller track the reference R?
- How much does it overshoot?
- Will the output eventually settle to the reference signal?
- How well does it compensate disturbances W to the plant?
- How sensitive is the controller to variations in the mathematical model?
- **How well does it ignore sensor noise?**
- **How many sensors are required and where to place them?**

Properties of Control

- How good does the controller track the reference R?
- How much does it overshoot?

} **tracking**

- Will the output eventually settle to the reference signal? } **stability**
- How well does it compensate disturbances W to the plant? } **regulation**

- How sensitive is the controller to variations in the mathematical model? } **sensitivity**

- How well does it ignore sensor noise? } **sensor fusion**

- How many sensors are required and where to place them? } **observability and state estimation**

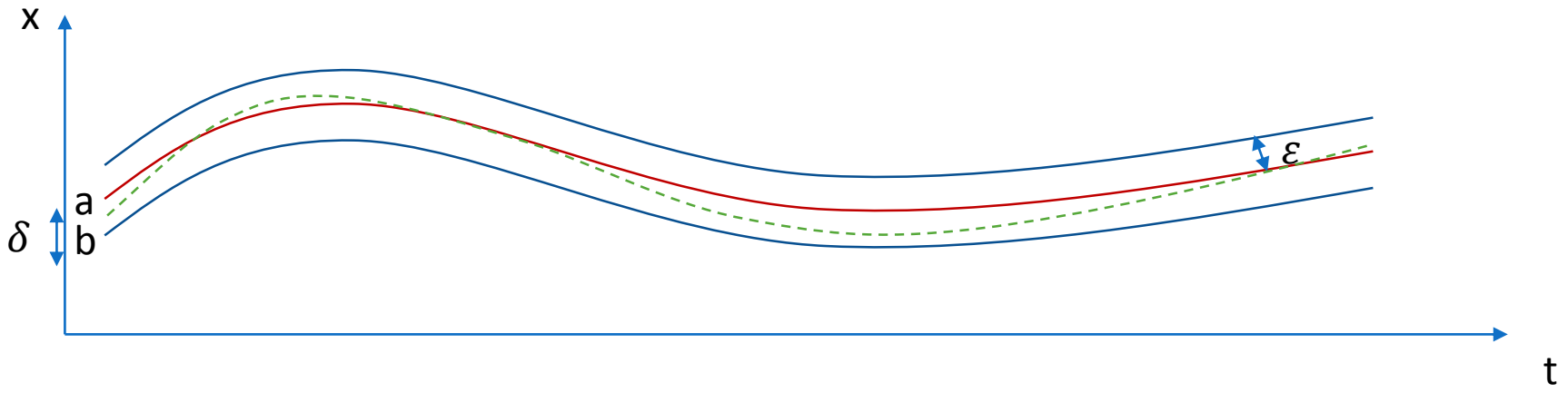
Stability

Determines whether or not solutions nearby the solution of a differential equation remain close, get closer or move further away.

Definition: Lyapunov Stability

Let $x(t;a)$ be a solution to the differential equation with initial condition a. A solution is stable if other solutions that start near a stay close to $x(t;a)$. Formally, we say that

$x(t;a)$ is *stable* if $\forall\, \varepsilon > 0. \exists\, \delta > 0. \|b - a\| < \delta \implies \|x(t;b) - x(t;a)\| < \varepsilon$

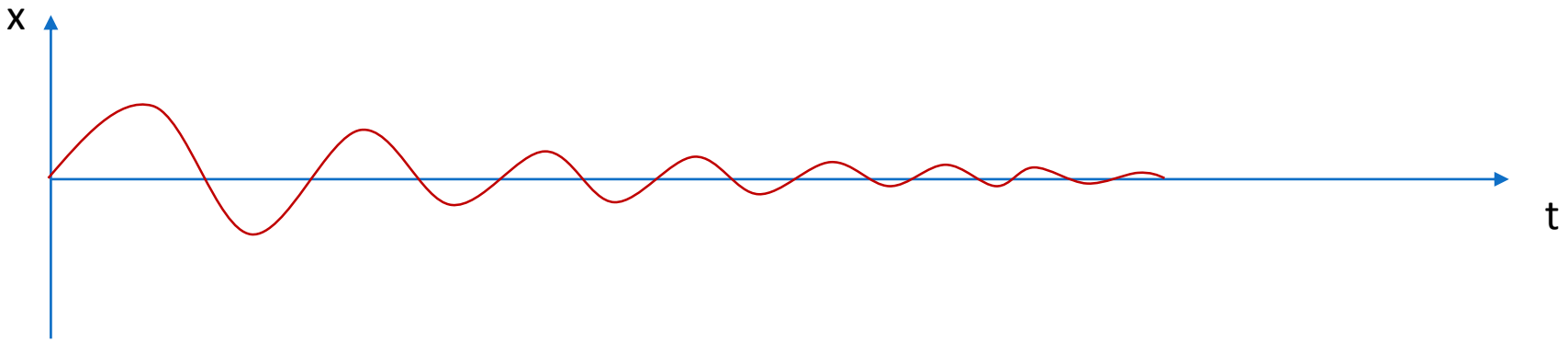


Asymptodic Stability

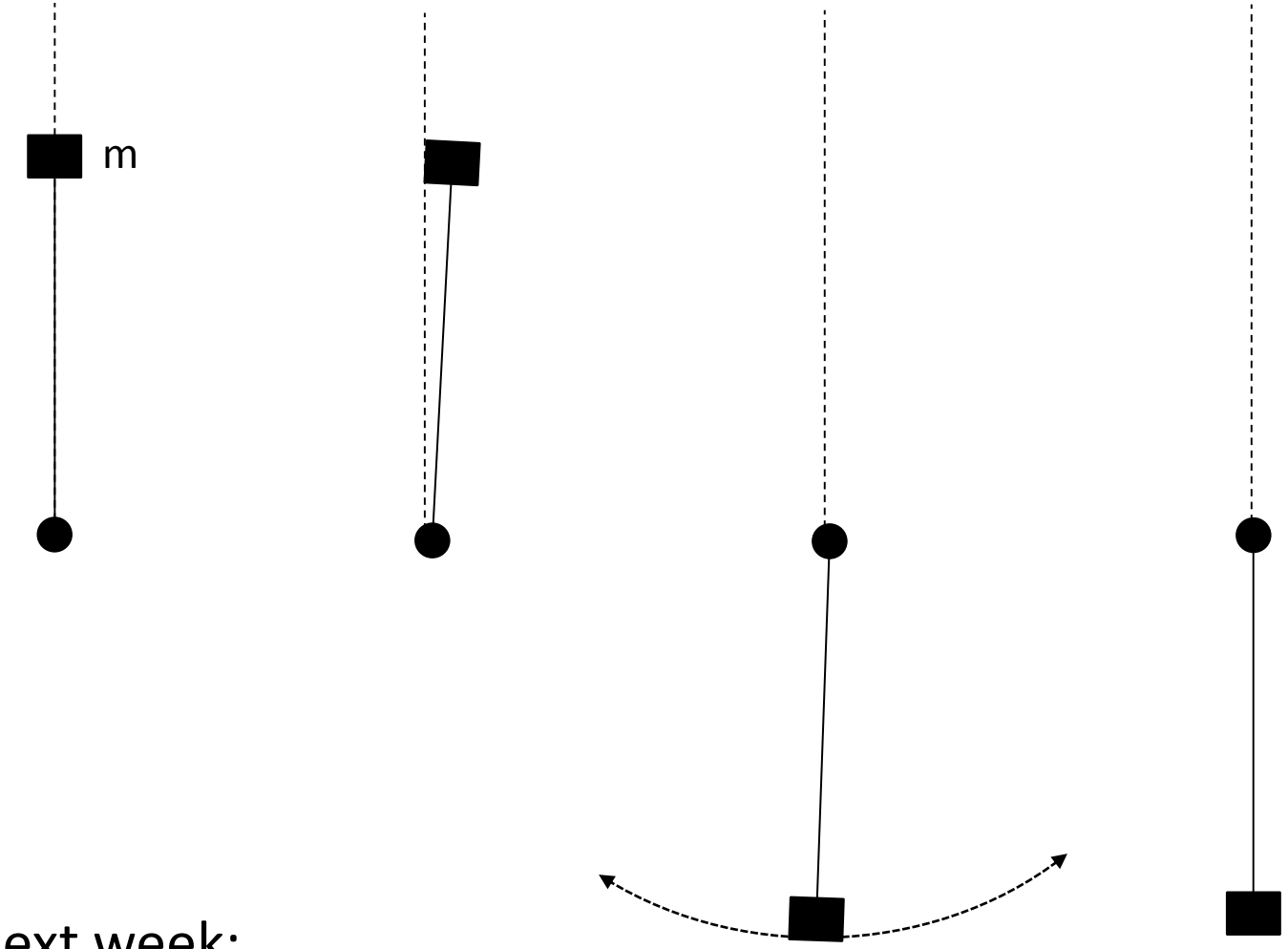
Convergence of solutions

Definition: Asymptodic Stability

A solution $x(t;a)$ is asymptodically stable if it is lyapunov stable and in addition $x(t;b) \to x(t;a)$ for t $\to \infty$.



Stability

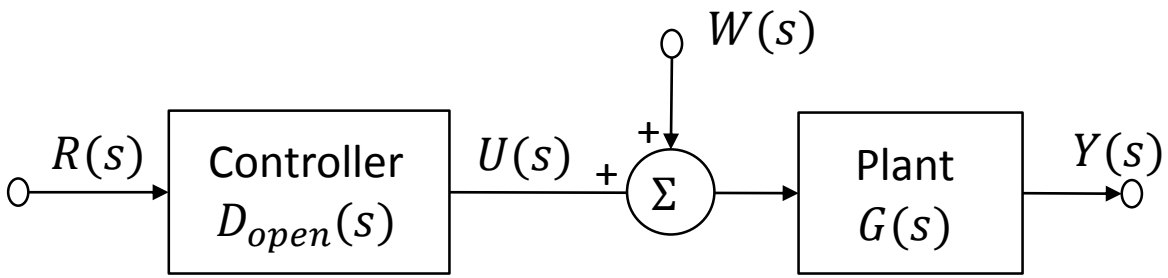


Next week:

- How to derive stability from location of zeros and poles in s-plane?
- How to move zeros and poles by adding control?

Tracking

Cause output to follow the reference as closely as possible



for open loop: $D_{open}(s) = \frac{1}{G(s)}$

but:
sensitivity

Exercise: Closed loop system with $G(s) = \frac{1}{s^2+3s+9}, D(s) = \frac{c_2 s^2 + c_1 s + c_0}{s(s+d_1)}$. Compute the coefficients c_i, d_i such that the closed loop has the characteristic equation $(s+6)(s+3)(s^2+3s+9) = 0$.
(characteristic equation is the dominator of the closed-loop transfer function set to zero)

Regulation

Keep the error small when the reference is at most a constant set point and disturbances are present.

Easy to see that open loop controller has no influence on disturbances.

Closed loop control: $E = SR - GSW + TV$ $\quad S = \frac{1}{1+GD}$ $\quad T = GDS = \frac{GD}{1+GD}$

- to minimize $GSW = \frac{G}{1+GD}W$ the controller term D must be as large as possible (ideally infinite).
- but controllers with a large D have no means to reduce sensor noise as

$$TV = \frac{GD}{1+GD}V$$

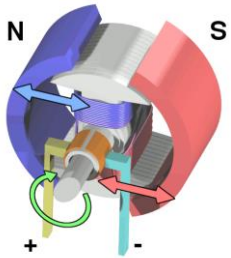
=> make D large for some frequencies but small for others!
disturbances of most plants occur at very low frequencies

Sensitivity

Suppose a plant is designed with gain G, how sensitive is the system to when this gain changes in operation to $G + \delta G$ (i.e., to a percent change of $\delta G / G$).

Next week: feedback control systems are much less error prone to plants changing their gain as open loop systems.

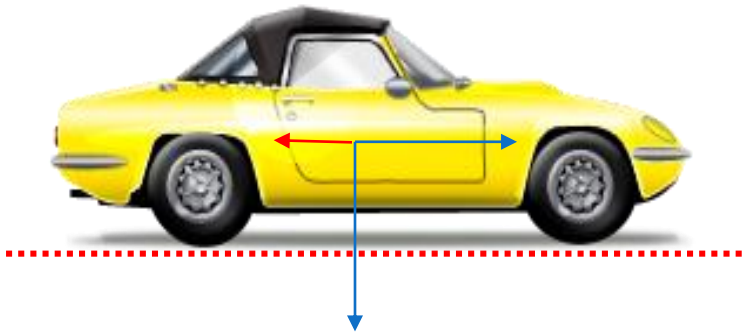
Observability



cc-by-sa 3 (Wapcaplet)

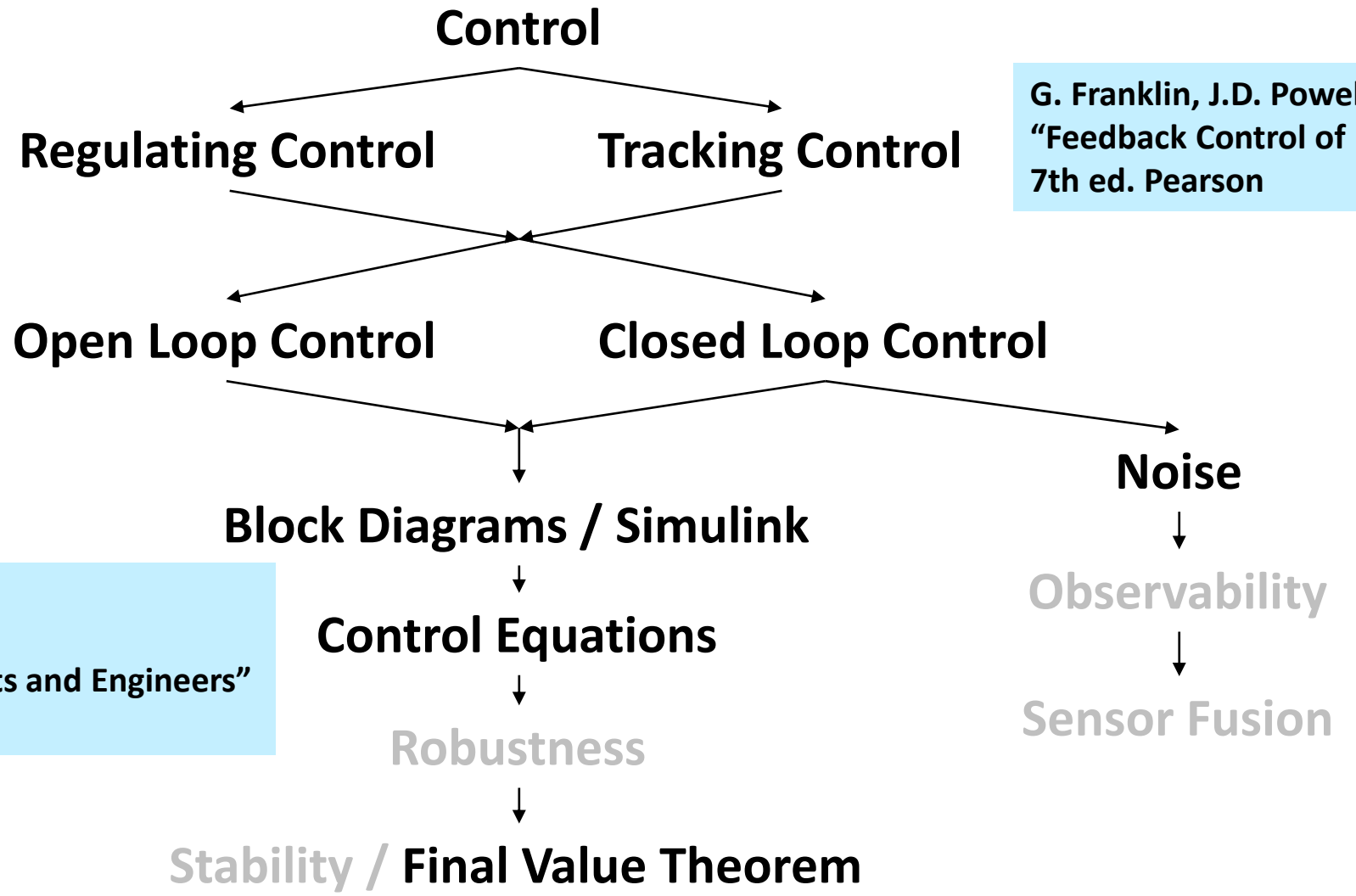
$$U = L\frac{\partial I}{\partial t} + RI + k_b\theta'$$
$$M\theta'' = k_T I - v\theta' - \tau$$

$$\tau = r \times F$$



$$v' + \frac{b}{m}v = \frac{F_0}{m}\partial(t)$$

Summary



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K. Åström, R. Murray
"Feedback Systems –
An Introduction for Scientists and Engineers"
Princeton University Press

Final Value Theorem

Suppose we have a laplace transform Y(s) of a signal y(t). Compute the steady-state value of y(t) from Y(s)!

One of 3 possible situations:

1) Signal contains exponential with positive coefficient
   => steady-state value will be unbounded as response grows over time.

2) Signal contains sinusoid (pairs of poles on imaginary axis of s-plane)
   => steady-state value will be undefined as sinusoid “wiggels”.

3) Signal contains exponentials with negative coefficients only (poles in left hand plane) except for one pole at s = 0
   => steady-state value will be the constant in time determined by s = 0 pole.

Final Value Theorem: If all poles of sY(s) are in the left half of the s-plane, then: $\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\,Y(s)$

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Control and Feedback **Transfer Functions and State Space Models**	Physics
PID Control Stability and Root Locust Method	Feedback Control
Mixed-Criticality Scheduling and Real-Time Operating Systems (RTOS)	RTOS
Program Verification Differential Dynamic Logic and KeYmaera X Differential Invariants	Verification