Constructing and Verifying Cyber Physical Systems
Control and Feedback

Marcus Völp
Laplace Transform

\[ \frac{d^2}{dt^2} e^{-i\omega t} = -\omega^2 e^{-i\omega t} \]

\[ \frac{d}{dt} e^{-\sigma t} = -\sigma e^{-\sigma t} \]

... \( \sigma = -1 \)  
premultiply exponential \( \sigma = 0 \)  
\( \sigma = 1 \) ...
Laplace Transform

\[ \mathcal{L}_f(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt \]

\( s = \sigma + i \omega \)
Laplace Transform

\[ V(s) = \frac{1}{m} \cdot \frac{b}{s + \frac{b}{m}} \cdot U(s) \]

\[ v(t) = v_\delta(t) \ast u(t) \]

\[ \mathcal{L}_-^{-1}(V)(t) \]
Exercise

\[ \mathcal{L}_-(f)(s) = \int_{0-}^{\infty} f(t) \, e^{-st} \, dt \]

**superposition**

\[ \mathcal{L}(af(t)+bg(t))(s) = \int_{0-}^{\infty} (af(t) + bg(t)) \, e^{-st} \, dt = a \int_{0-}^{\infty} f(t) \, e^{-st} \, dt + b \int_{0-}^{\infty} g(t) \, e^{-st} \, dt = a \mathcal{L}(f)(s) + b \mathcal{L}(g)(s) \]

**time delay**

\[ \mathcal{L}(f(t-\lambda))(s) = \int_{0-}^{\infty} f(t-\lambda) \, e^{-st} \, dt = \int_{0-}^{\infty} f(x) \, e^{-s(x+\lambda)} \, dx = e^{-s\lambda} \int_{0-}^{\infty} f(x) \, e^{-sx} \, dx = e^{-s\lambda} \mathcal{L}(f)(s) \]

**frequency shift**

\[ \mathcal{L}(e^{-at}f(t))(s) = \int_{0-}^{\infty} e^{-at} f(t) \, e^{-st} \, dt = \int_{0-}^{\infty} f(t) \, e^{-(s+a)t} \, dt = \mathcal{L}(f)(s+a) \]

**convolution**

\[ \mathcal{L}(f(t) \ast g(t))(s) = \int_{0-}^{\infty} \int_{0-}^{\infty} f(\tau)g(t-\tau) \, d\tau \, e^{-st} \, dt = \int_{0-}^{\infty} f(\tau) \left( \int_{0-}^{\infty} g(\sigma) \, e^{-s\sigma} \, d\sigma \right) \, e^{-s\tau} \, d\tau = \mathcal{L}(f)(s) \mathcal{L}(g)(s) \]
Overview

Introduction
Mathematical Foundations (Differential Equations and Laplace Transformation)

Control and Feedback

Transfer Functions and State Space Models

PID Control
Stability and Root Locust Method

Mixed-Criticality Scheduling and Real-Time Operating Systems (RTOS)

Program Verification
Differential Dynamic Logic and KeYmaera X
Differential Invariants
Overview

Control

Regulating Control  Tracking Control

Open Loop Control  Closed Loop Control

Block Diagrams / Simulink

Control Equations

Robustness

Stability / Final Value Theorem

Noise

Observability

Sensor Fusion

K. Åström, R. Murray
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Control: adhere to reference value

- **Regulating Control (Regulator)**
  Maintain desired output despite external disturbances.

- **Tracking Control (Servo)**
  Follow reference.

\[ v \rightarrow v_{\text{desired}} \]
Feedback

- **Open Loop Control / Feedforward Control**
  no measurement of output / no correction to make output conform to reference signal

- **Closed Loop Control / Feedback Control**
  adjust input based on measured output
Modeling Concepts

- Heritage of Mechanics
  Describe the evolution of the system state with ordinary differential equations

\[ U = L \frac{\partial I}{\partial t} + RI + k_b \theta' \]

\[ M \theta'' = k_T I - \nu \theta' - \tau \]

\[ \tau = r \times F \]

\[ \frac{T(s)}{U(s)} = \frac{k_T}{L} \frac{1}{s + \frac{R}{L}} \]

\[ \nu' + \frac{b}{m} \nu = \frac{F_0}{m} \partial(t) \]

\[ \tau = \frac{\tau}{F} \]

\[ V(s) \frac{1}{U(s)} = \frac{1}{s + \frac{b}{m}} \]
Modeling Concepts

- Heritage of Electrical Engineering
  Describe the input/output behavior of connected components

Resistor
\[ u = Ri \]

Capacitor
\[ i = C \frac{du}{dt} \]

Inductor
\[ u = L \frac{di}{dt} \]

Current source
\[ i = i_s \]

Voltage source
\[ u = u_s \]

Kirchhoff’s current law (KCL):
sum of currents leaving a junction or node equals sum of currents entering the node

Kirchhoff’s voltage law (KVL):
sum of voltages taken around a closed path in a circuit is zero
Modeling Concepts

- Heritage of Electrical Engineering
  Describe the input/output behavior of connected components

\[ i_1 = C_1 \frac{\partial u_1}{\partial t} \]
\[ i_2 = C_2 \frac{\partial u_2}{\partial t} \]
\[ i_3 = \frac{u_1}{R_1} \]
\[ i_4 = \frac{u_2}{R_2} \]
\[ u_1 - u_2 = L \frac{\partial i_L}{\partial t} \]

\[ i(t) = i_3 + i_1 + i_L \]
\[ i_L = i_2 + i_4 \]
Modeling Concepts

- Heritage of Electrical Engineering
  Describe the input/output behavior of connected components

\[
V(s) = \frac{1}{m} \frac{1}{s + \frac{b}{m}}
\]

Diagram:
- Controller
- Actuator
- DC motor
- Plant
- Sensor
- Desired velocity \(v_{desired}\)
- Measured velocity \(v_{measured}\)
- Input \(I\)
- Force \(F\)
- Velocity \(v\)
MatLab / Simulink
Block Diagrams

Open Loop Structures

\[
\frac{Y_2(s)}{U_1(s)} = G_2 G_1
\]

Closed Loop Structures

\[
\begin{align*}
U_1(s) &= R(s) - Y_2(s) \\
Y_2(s) &= G_2(s) G_1(s) U_1(s) \\
Y_1(s) &= G_1(s) U_1(s) \\
Y_1(s) &= \frac{G_1(s)}{1 + G_1(s) G_2(s)} R(s)
\end{align*}
\]
Basic Equations of Control

Y = GDR + GW

error

\[ E = R - Y = [1 - GD]R - GW \]

- How good does the controller track the reference R?
- How much does it overshoot?
- Will the output eventually settle to the reference signal?
- How well does it compensate disturbances W to the plant?
- How sensitive is the controller to variations in the mathematical model?
Basic Equations of Control

Closed Loop Control

\[ Y = TR + GSW - TV \]

\[ S = \frac{1}{1 + GD} \]

\[ T = GDS = \frac{GD}{1 + GD} \]

- How good does the controller track the reference R?
- How much does it overshoot?
- Will the output eventually settle to the reference signal?
- How well does it compensate disturbances W to the plant?
- How sensitive is the controller to variations in the mathematical model?
- **How well does it ignore sensor noise?**
- How many sensors are required and where to place them?
Properties of Control

- How good does the controller track the reference $R$?  \[ \{ \text{tracking} \]  
- How much does it overshoot?  
- Will the output eventually settle to the reference signal?  \[ \{ \text{stability} \]  
- How well does it compensate disturbances $W$ to the plant?  \[ \{ \text{regulation} \]  
- How sensitive is the controller to variations in the mathematical model?  \[ \{ \text{sensitivity} \]  
- How well does it ignore sensor noise?  \[ \{ \text{sensor fusion} \]  
- How many sensors are required and where to place them?  \[ \{ \text{observability and state estimation} \]
Determines whether or not solutions nearby the solution of a differential equation remain close, get closer or move further away.

**Definition: Lyapunov Stability**

Let $x(t; a)$ be a solution to the differential equation with initial condition $a$. A solution is stable if other solutions that start near a stay close to $x(t; a)$. Formally, we say that

$x(t; a)$ is stable if $\forall \varepsilon > 0. \exists \delta > 0. \|b - a\| < \delta \implies \|x(t; b) - x(t; a)\| < \varepsilon$
Asymptodic Stability

Convergence of solutions

**Definition: Asymptodic Stability**

A solution $x(t; a)$ is asymptodically stable if it is lyapunov stable and in addition

$x(t; b) \to x(t; a)$ for $t \to \infty$. 

![Graph showing convergence of solutions](image)
Next week:
• How to derive stability from location of zeros and poles in s-plane?
• How to move zeros and poles by adding control?
Tracking

Cause output to follow the reference as closely as possible

\[
\begin{align*}
R(s) & \xrightarrow{D_{\text{open}}(s)} U(s) \xrightarrow{\Sigma} Y(s) \\
& + \rightarrow W(s)
\end{align*}
\]

For open loop: \(D_{\text{open}}(s) = \frac{1}{G(s)}\)

**but:** sensitivity

Exercise: Closed loop system with \(G(s) = \frac{1}{s^2 + 3s + 9}\), \(D(s) = \frac{c_2s^2 + c_1s + c_0}{s(s + d_1)}\). Compute the coefficients \(c_i, d_i\) such that the closed loop has the characteristic equation \((s + 6)(s + 3)(s^2 + 3s + 9) = 0\). (characteristic equation is the dominator of the closed-loop transfer function set to zero)
Keep the error small when the reference is at most a constant set point and disturbances are present.

Easy to see that open loop controller has no influence on disturbances.

Closed loop control:

\[ E = SR - GSW + TV \]
\[ S = \frac{1}{1 + GD} \]
\[ T = GDS = \frac{GD}{1 + GD} \]

- to minimize \( GSW = \frac{G}{1 + GD} W \) the controller term \( D \) must be as large as possible (ideally infinite).
- but controllers with a large \( D \) have no means to reduce sensor noise as

\[ TV = \frac{GD}{1 + GD} V \]

=> make \( D \) large for some frequencies but small for others!

disturbances of most plants occur at very low frequencies.
Suppose a plant is designed with gain $G$, how sensitive is the system to when this gain changes in operation to $G + \delta G$ (i.e., to a percent change of $\delta G/G$).

Next week: feedback control systems are much less error prone to plants changing their gain as open loop systems.
Observability

\[ U = L \frac{\partial I}{\partial t} + RI + k_b \theta' \]

\[ M \theta'' = k_T I - \nu \theta' - \tau \]

\[ \tau = \tau \times F \]

\[ \nu' + \frac{b}{m} \nu = \frac{F_0}{m} \partial(t) \]
Summary

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Final Value Theorem

Suppose we have a laplace transform $Y(s)$ of a signal $y(t)$. Compute the steady-state value of $y(t)$ from $Y(s)$!

One of 3 possible situations:

1) Signal contains exponential with positive coefficient
   $\Rightarrow$ steady-state value will be unbounded as response grows over time.

2) Signal contains sinusoid (pairs of poles on imaginary axis of s-plane)
   $\Rightarrow$ steady-state value will be undefined as sinusoid “wiggels”.

3) Signal contains exponentials with negative coefficients only (poles in left hand plane) except for one pole at $s = 0$
   $\Rightarrow$ steady-state value will be the constant in time determined by $s = 0$ pole.

Final Value Theorem: If all poles of $sY(s)$ are in the left half of the s-plane, then:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} s Y(s)$$
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<table>
<thead>
<tr>
<th>Introduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematical Foundations (Differential Equations and Laplace Transformation)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
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</tr>
<tr>
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</tr>
</tbody>
</table>

<table>
<thead>
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</tr>
</thead>
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<thead>
<tr>
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</tr>
</thead>
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</tr>
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