TECHNISCHE UNIVERSITÄT DRESDEN

Constructing and Verifying Cyber Physical Systems

Differential Invariants

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Overview

Introduction	Math
Mathematical Foundations (Differential Equations and Laplace Transformation)	
Control and Feedback	Physics
Transfer Functions and State Space Models	
Poles, Zeros / PID Control	Feedback Control
Stability, Root Locust Method, Digital Control	
Mixed-Criticality Scheduling and Real-Time Operating Systems (RTOS)	RTOS
Coordinating Networked Cyber-Physical Systems	CPS
Program Verification	Verification
Differential Dynamic Logic and KeYmaera X	
Differential Invariants	

Overview

Differential Algebraic Logic

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symbolaris.com

Local Reasoning about Differential Equations

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Andre Platzer: Logical Analysis of Hybrid-Systems

Differential Cut / Weakening

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Differential Induction and Differential Invariants

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Proving While Termination

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Differential Variants

↓

How to improve this Course?

Differential Dynamic Logic

$$(\text{i}\forall \text{ quantifier elimination})\ \frac{\Gamma \vdash \mathrm{QE}(\forall X\,(\Phi(X) \vdash \Psi(X))), \Delta}{\Gamma, \Phi(s(X_1,..,X_n)) \vdash \Psi(s(X_1,..,X_n)), \Delta}\ 3 \qquad (\text{i}\exists \text{ eliminate existential})\ \frac{\Gamma \vdash \mathrm{QE}(\exists X\ \bigwedge_i(\Phi_i \vdash \Psi_i)), \Delta}{\Gamma, \Phi_1 \vdash \Psi_1, \Delta \quad \ldots \quad \Gamma, \Phi_n \vdash \Psi_n, \Delta}\ 4$$

$$([]\text{ generalization})\ \frac{\Gamma \vdash [\alpha]\phi, \Delta \qquad \Gamma \vdash \forall^\alpha(\phi \to \psi), \Delta}{\Gamma \vdash [\alpha]\psi, \Delta} \qquad (\langle\rangle\text{ generalization})\ \frac{\Gamma \vdash \langle\alpha\rangle\phi, \Delta \qquad \Gamma \vdash \forall^\alpha(\phi \to \psi), \Delta}{\Gamma \vdash \langle\alpha\rangle\psi, \Delta}$$

$$(\textit{ind}\text{ loop invariant})\ \frac{\Gamma \vdash \phi, \Delta \qquad \Gamma \vdash \forall^\alpha(\phi \to [\alpha]\phi), \Delta \qquad \Gamma \vdash \forall^\alpha(\phi \to \psi), \Delta}{\Gamma \vdash [\alpha^*]\psi, \Delta}$$

$$(\textit{con}\text{ loop convergence})\ \frac{\Gamma \vdash \exists v\,\varphi(v), \Delta \qquad \Gamma \vdash \forall^\alpha \forall v{>}0\,(\varphi(v) \to \langle\alpha\rangle\varphi(v-1)), \Delta \qquad \Gamma \vdash \forall^\alpha(\exists v{\leq}0\,\varphi(v) \to \psi), \Delta}{\Gamma \vdash \langle\alpha^*\rangle\psi, \Delta}$$

$$(DI\text{ differential invariant})\ \frac{\Gamma, H \vdash F, \Delta \qquad \Gamma \vdash \forall^\alpha(H \to F'^{\theta_1 \ldots \theta_n}_{x_1' \ldots x_n'}), \Delta}{\Gamma \vdash [x_1' = \theta_1, .., x_n' = \theta_n \,\&\, H]F, \Delta}$$

$$(DV\text{ differential variant})\ \frac{\Gamma \vdash [x_1' = \theta_1, .., x_n' = \theta_n \,\&\, {\sim}F]H, \Delta \qquad \Gamma \vdash \exists\varepsilon{>}0\,\forall^\alpha(\neg F \wedge H \to (F' \geq \varepsilon)^{\theta_1 \ldots \theta_n}_{x_1' \ldots x_n'}), \Delta}{\Gamma \vdash \langle x_1' = \theta_1, .., x_n' = \theta_n \,\&\, H\rangle F, \Delta}\ 5$$

$$(DW\text{ differential weaken})\ \frac{\Gamma \vdash \forall^\alpha(H \to \phi), \Delta}{\Gamma \vdash [x' = \theta \,\&\, H]\phi, \Delta}$$

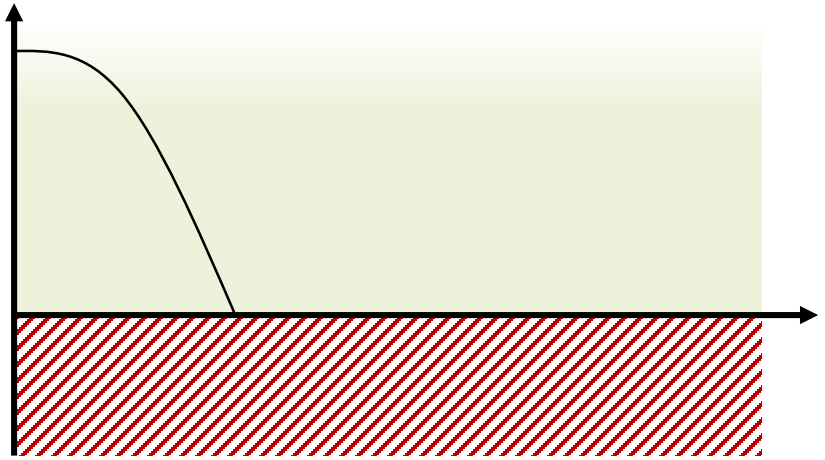
$$(DC\text{ differential cut})\ \frac{\Gamma \vdash [x' = \theta \,\&\, H]C, \Delta \qquad \Gamma \vdash [x' = \theta \,\&\, (H \wedge C)]\phi, \Delta}{\Gamma \vdash [x' = \theta \,\&\, H]\phi, \Delta}$$

$$(DA\text{ differential auxiliaries})\ \frac{\phi \leftrightarrow \exists y\,\psi \qquad \Gamma \vdash [x' = \theta, y' = \vartheta \,\&\, H]\psi, \Delta}{\Gamma \vdash [x' = \theta \,\&\, H]\phi, \Delta}\ 6$$

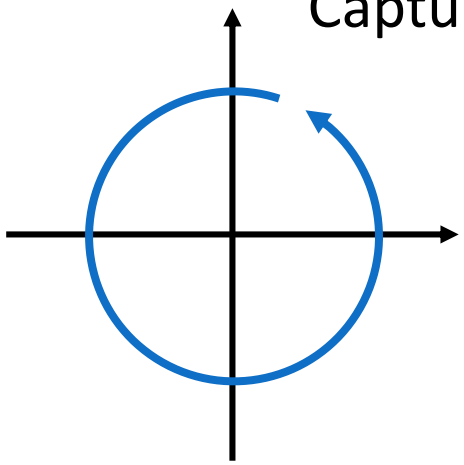
$$(IA\text{ auxiliary variable})\ \frac{\Gamma \vdash [y := \theta]\phi, \Delta}{\Gamma \vdash \phi, \Delta}\ 7 \qquad (\langle :* \rangle\text{ random})\ \frac{\exists X\,\langle x := X\rangle\phi}{\langle x := *\rangle\phi}\ 8 \qquad ([:*]\text{ random})\ \frac{\forall X\,[x := X]\phi}{[x := *]\phi}\ 8$$

Differential Algebraic Logic

$$\{h' = v, v' = -g \,\&\, h \geq 0\}$$



Capture short term evolution of complicated dynamics

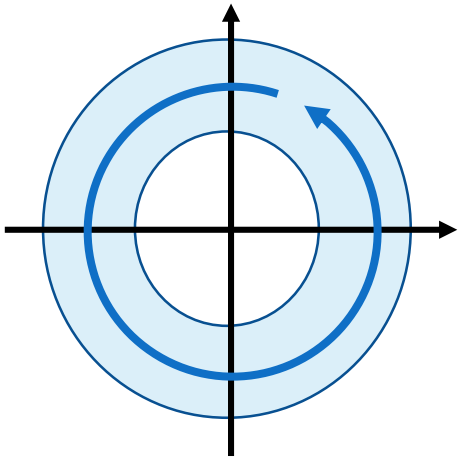


$$\{x' = r\,cos(\varphi), y' = r\,sin(\varphi), \varphi' = \omega\}$$

$$\{x' = d_1, y' = d_2 \,\&\, d_1' = -\omega d_2 \wedge d_2' = \omega d_1 \wedge d_1^2 + d_2^2 = r^2\}$$

Differential Algebraic Logic

$$\{h' = v, v' = -g \,\&\, h \geq 0\}$$



Capture parametric systems with reasonable bounds for all parameters

=> we need a symbolic solution

$$\{x' = d_1, y' = d_2 \,\&\, \exists\omega.\, -1 < \omega \wedge w < 1 \wedge d_1' = -\omega d_2 \wedge d_2' = \omega d_1 \wedge d_1^2 + d_2^2 = r^2\}$$

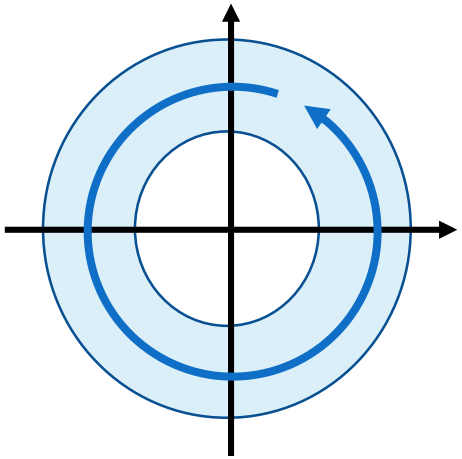
$$\{x' = d_1, y' = d_2 \,\&\, \exists r.\, 3 < r \wedge r < 4 \wedge d_1' = -\omega d_2 \wedge d_2' = \omega d_1 \wedge d_1^2 + d_2^2 = r^2\}$$

Differential Algebraic Constraints

E.g. "Whatever parametrized obstacles do, the car has a chance to avoid collision!" $\exists p.\, [\alpha]\langle\beta\rangle\varphi$

Differential Jump Constraints: $\exists a.\, (w := a^2 \wedge a < 5)$

Differential Algebraic Logic



Semantic:

Differential Jump Constraints:

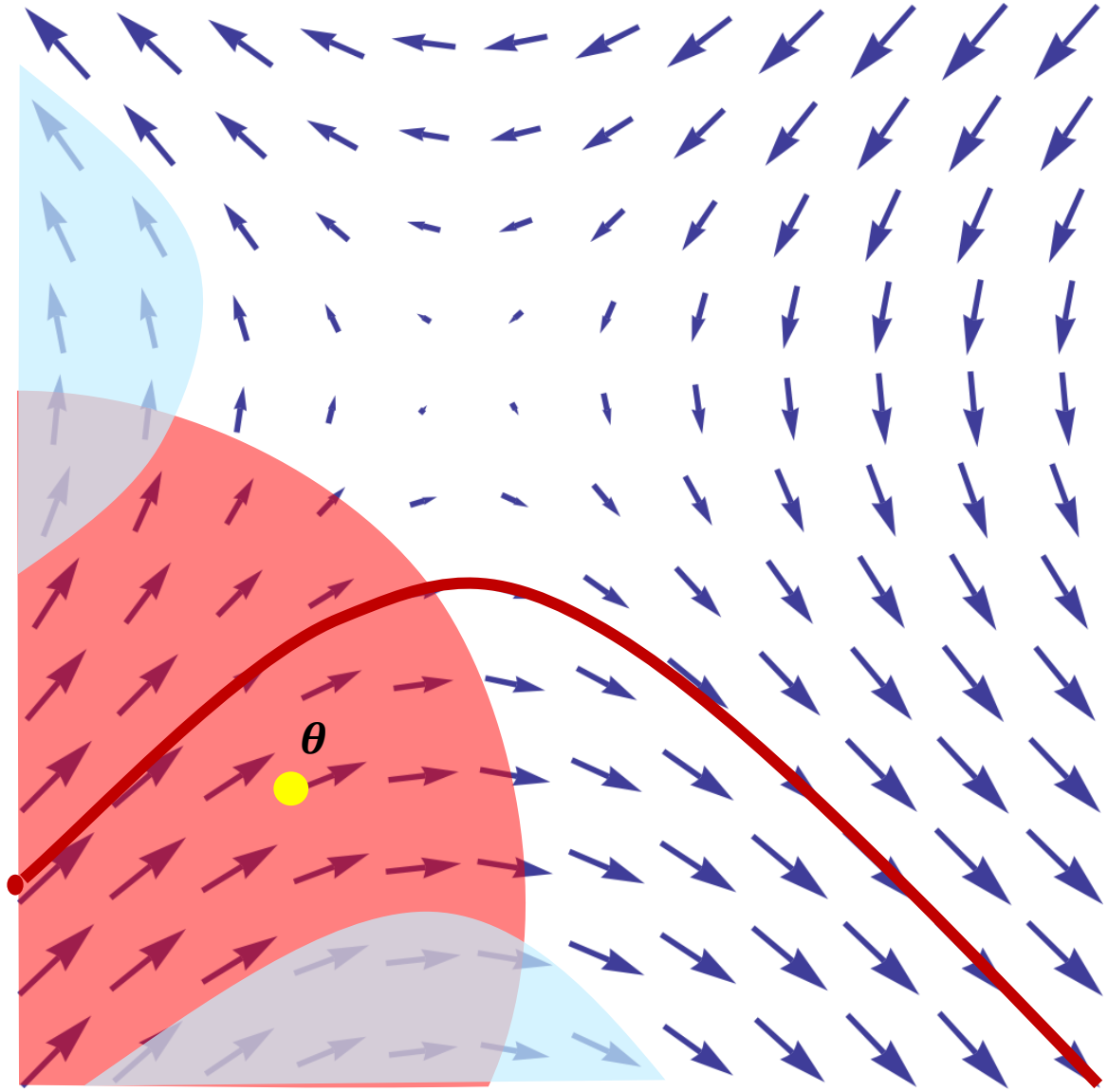
$(v, w) \vDash \forall x.\, \Phi \iff (v_x, w) \vDash \Phi$

for all states v_x that agree with v but have an arbitrary value r at x

$(v, w) \vDash \exists x.\, \Phi \iff (v_x, w) \vDash \Phi$

for some v_x with some value r at x

Local Reasoning about Diff. Eq.



$$\{x' = \theta \,\&H\}$$

At every point, the right hand side of the differential equation describes the dynamics where this point moves to.

$(DI$ differential invariant$)$

$(DV$ differential variant$)$

$(DW$ differential weaken$)$

$(DC$ differential cut$)$

$(DA$ differential auxiliaries$)$

Symbolic Differentiation

Idea: compute the derivative of the differential equation and the property to show
=> $\boldsymbol{\theta}$ in $\{\boldsymbol{x}' = \boldsymbol{\theta}\}$ immediately reveals how the property evolves at each point

$$D(r) = 0$$

$$D\left(x^{(n)}\right) = x^{(n+1)} \quad (e.g., D(x') = x'')$$

$$D(a + b) = D(a) + D(b)$$

$$D(a - b) = D(a) - D(b)$$

$$D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$$

$$D\left(\frac{a}{b}\right) = D(a) \cdot b - a \cdot D(b)/b^2$$

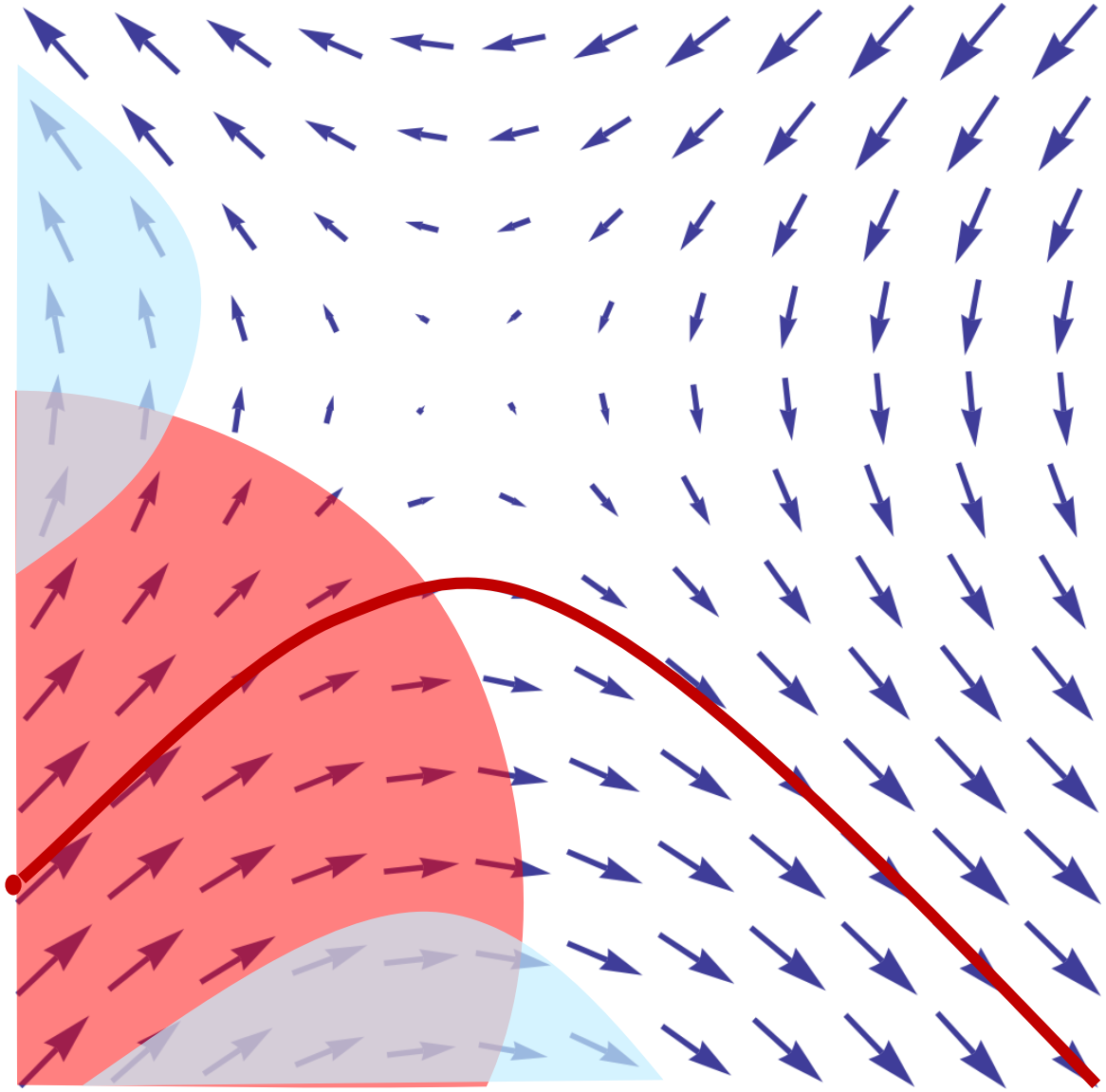
$$D(F) = \bigwedge_{i=1}^{m} D(F_i) \; where \; \{F_1, \ldots, F_m\} \; is \; the \; set \; of \; literals \; in \; F$$

$$D(a * b) = D(a) * D(b) \; where \; * \; \epsilon \{\leq, \geq, <, >, =\}$$

$$D(\neg a) = \neg D(a)$$

in particular: $D(a \vee b) = D(a) \wedge D(b)$

Local Reasoning about Diff. Eq.



(DI differential invariant)

(DV differential variant)

(DW differential weaken)

(DC differential cut)

(DA differential auxiliaries)

differentially augmented state

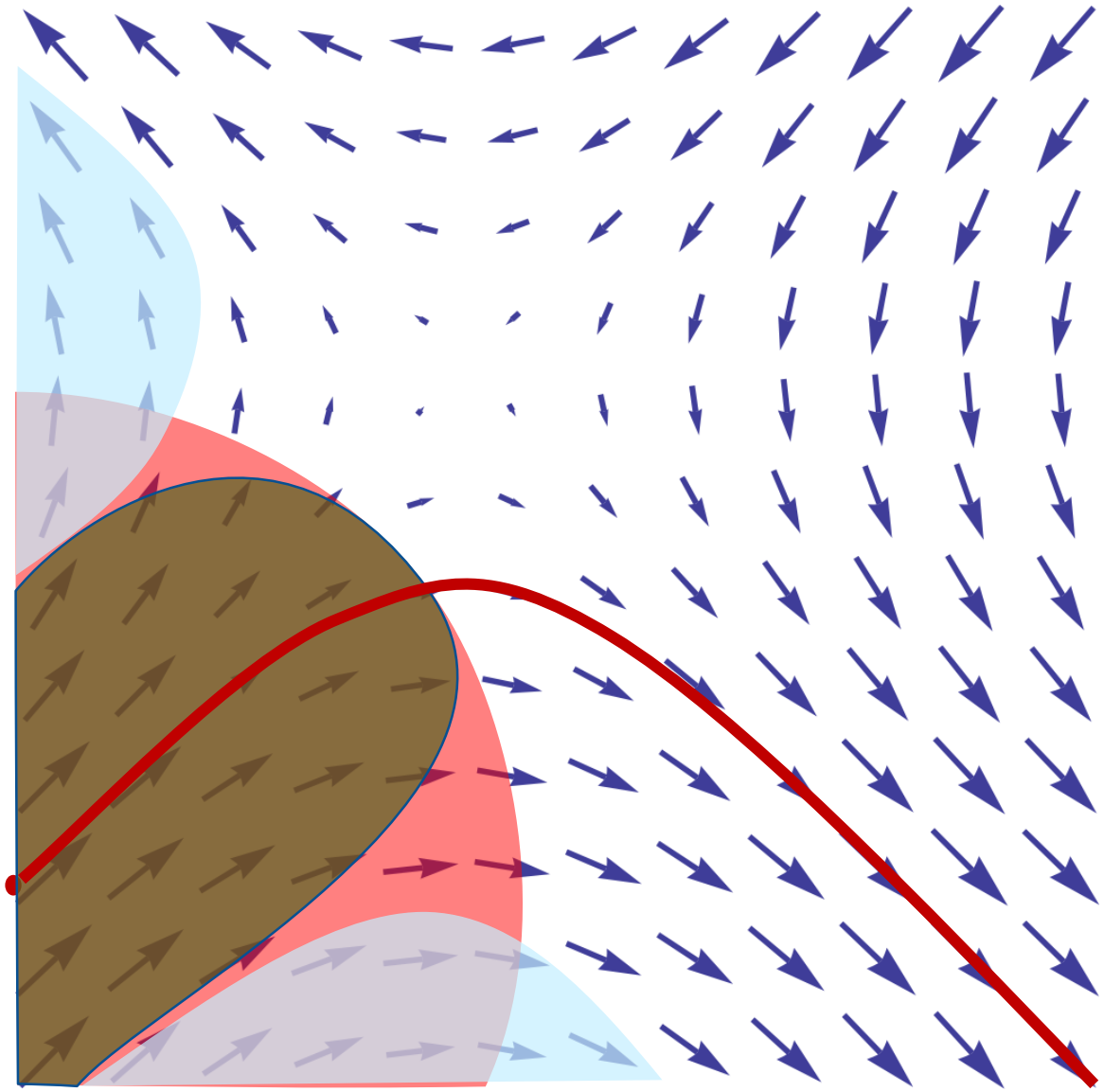
like $\varphi(\zeta)$ except $x^{(n)} = \frac{d^n \varphi(t)(x)}{dt^n}(\zeta)$

Lemma 3.1: (Derivation Lemma).

$$\frac{\boldsymbol{d\, val(\varphi(t), \theta)}}{\boldsymbol{dt}}(\boldsymbol{\zeta}) = \boldsymbol{val}\big(\overline{\boldsymbol{\varphi}}(\boldsymbol{\zeta}), \boldsymbol{D(\theta)}\big)$$

along the flow, analytic derivatives of valuations co-incide with valuations of syntactic derivations.

Differential Weaken



(DI differential invariant)

(DV differential variant)

(DW differential weaken)

(DC differential cut)

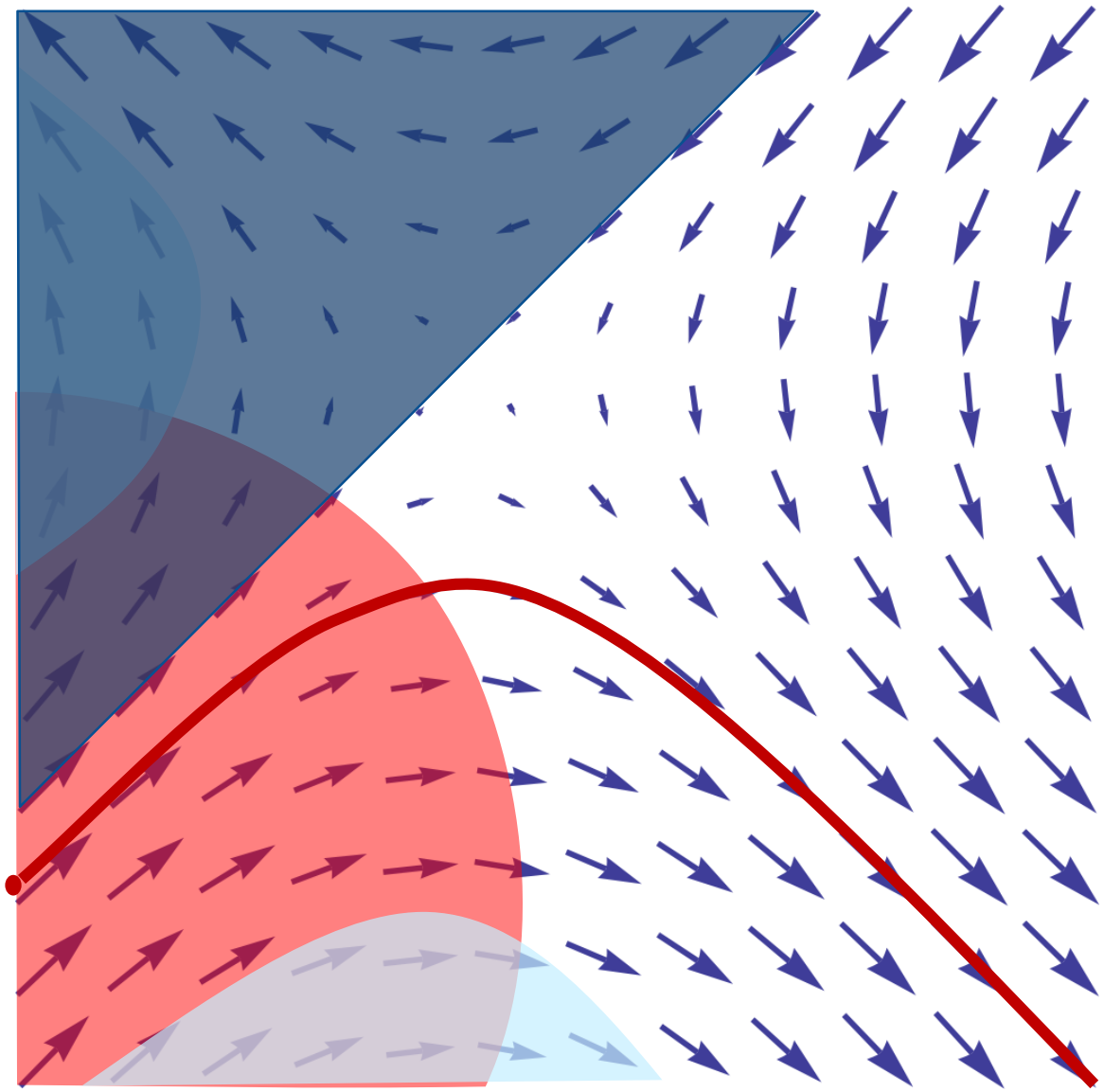
(DA differential auxiliaries)

DW: differential weaken

$$\frac{\Gamma \vdash \forall^{\alpha}(H \rightarrow \phi), \Delta}{\Gamma \vdash [x' = \theta \& H]\phi, \Delta}$$

Ignore differential equation if evolution domain constraint H already implies ϕ.

Differential Cut



(DI differential invariant)

(DV differential variant)

(DW differential weaken)

(DC differential cut)

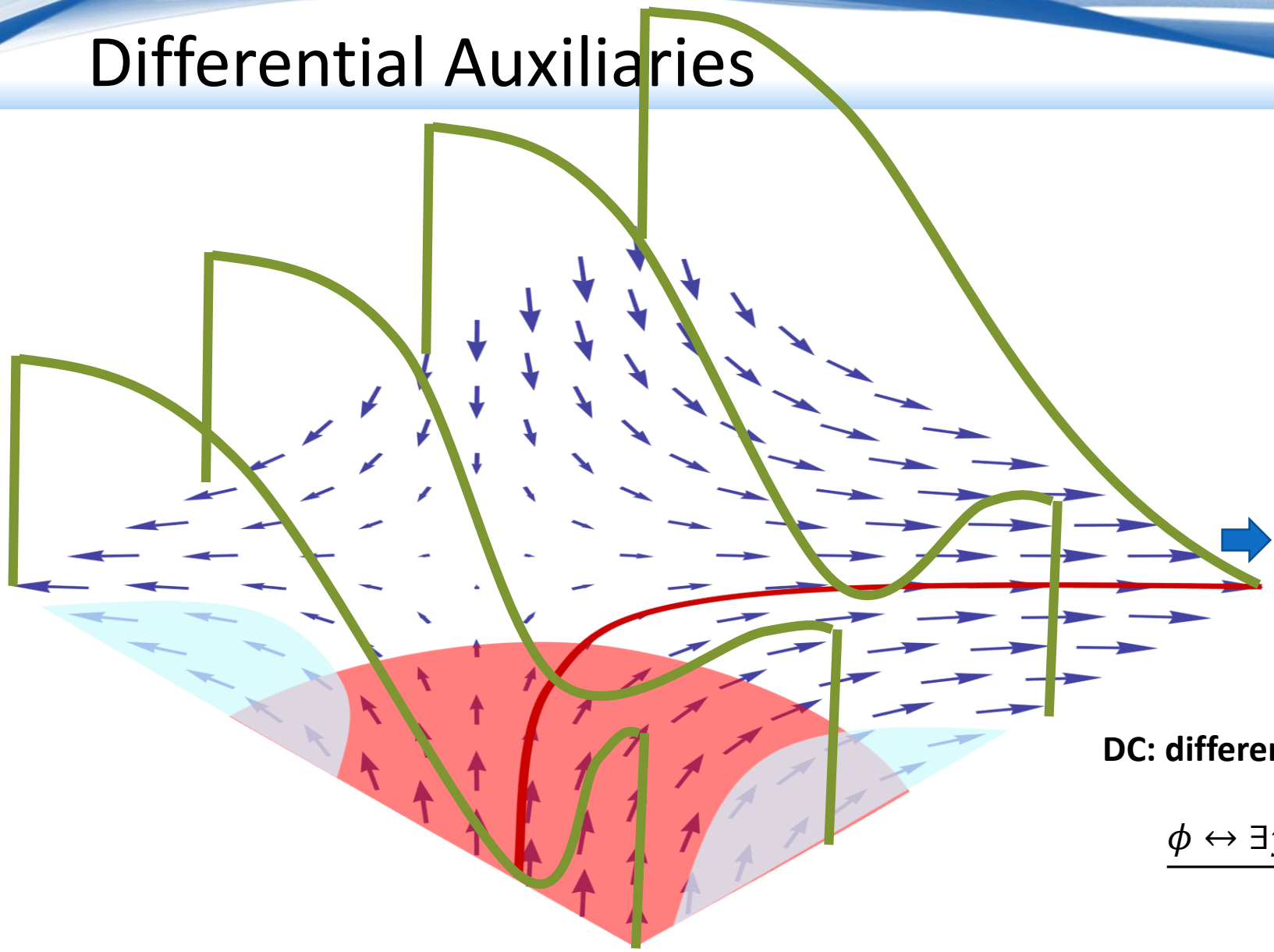
(DA differential auxiliaries)

DC: differential cut

$$\frac{\Gamma \vdash [x' = \theta \& H]C, \Delta \qquad \Gamma \vdash [x' = \theta \& H \wedge C]\phi, \Delta}{\Gamma \vdash [x' = \theta \& H]\phi, \Delta}$$

Ignore differential equation if evolution domain constraint H already implies ϕ.

Differential Auxiliaries



(DI differential invariant)

(DV differential variant)

(DW differential weaken)

(DC differential cut)

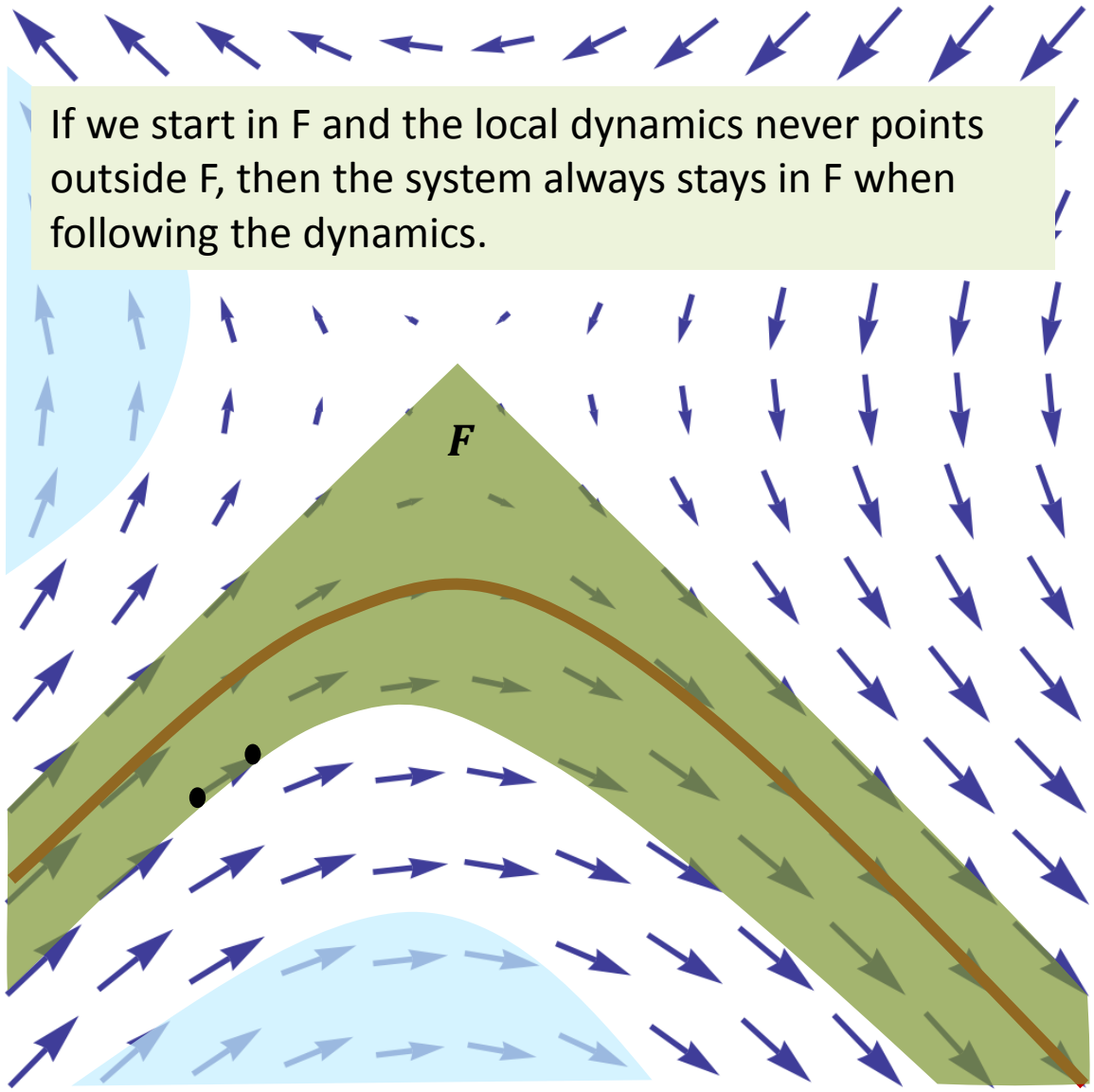
(DA differential auxiliaries)

DC: differential cut

$$\frac{\phi \leftrightarrow \exists y \Psi \qquad \Gamma \vdash [x' = \theta, y' = \nu \& H]\Psi, \Delta}{\Gamma \vdash [x' = \theta \& H]\phi, \Delta}$$

Ignore differential equation if evolution domain constraint H already implies ϕ.

Differential Induction



Induction (over iterations)

$$\frac{\vdash \forall^{\alpha} \phi \rightarrow [\alpha]\phi}{\phi \vdash [\alpha^*]\phi}$$

DI: Differential Induction

- Continuous form of induction
- Derivative points to the direction how the solution of a differential equation will evolve

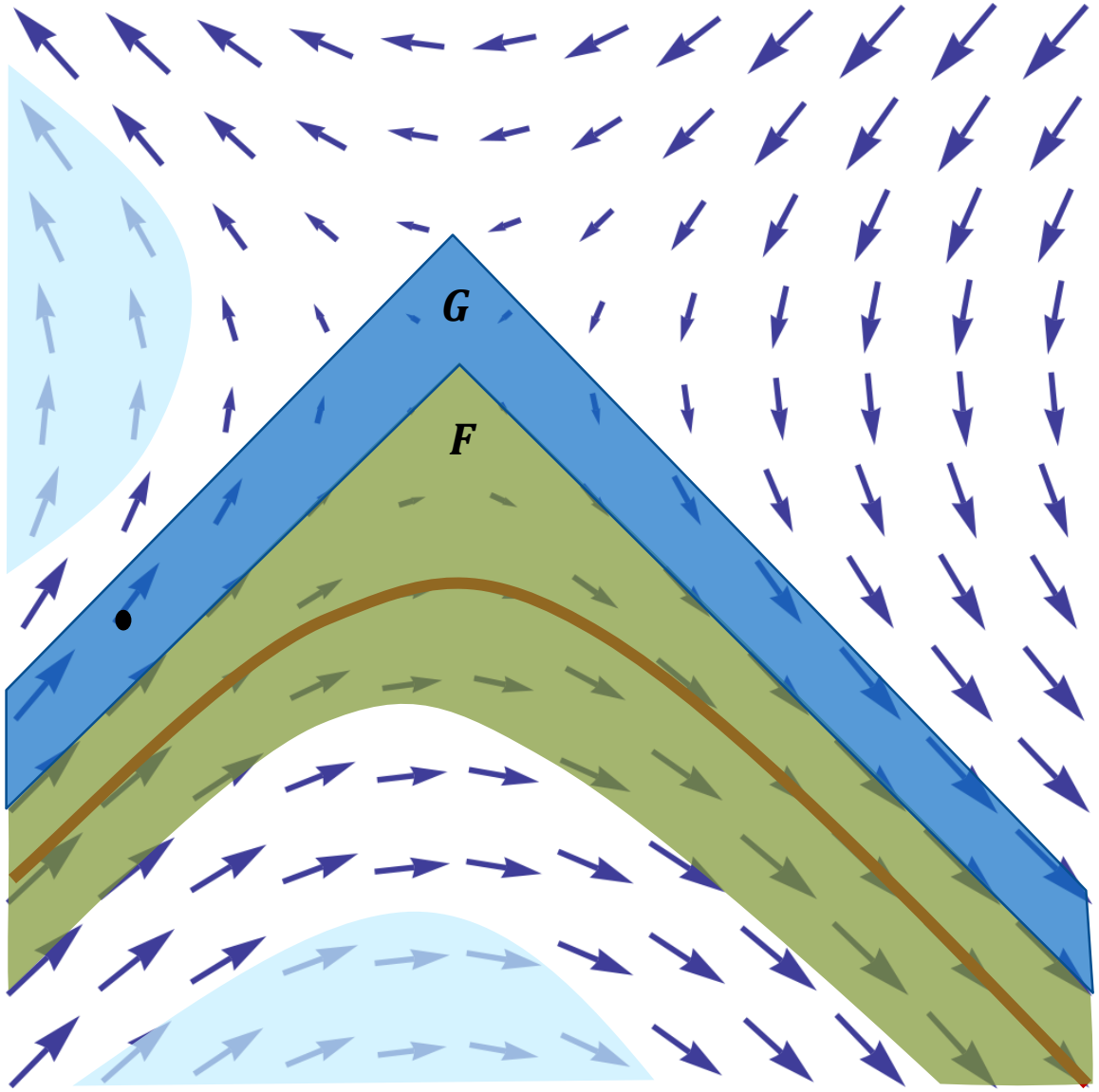
$$\frac{\Gamma, H \vdash F, \Delta \qquad \Gamma \vdash \forall^{\alpha}(H \rightarrow {F'}^{\theta_1,\ldots,\theta_n}_{x'_1,\ldots,x'_n}), \Delta}{\Gamma \vdash [x'_1 = \theta_1, \ldots, x'_n = \theta_n \& H]F, \Delta}$$

where ${F'}^{\theta_1,\ldots,\theta_n}_{x'_1,\ldots,x'_n}$ abbreviates $D(F)$ with $z' = 0$ for variables that do not change.

Differential Invariant:

- Region where the derivative always points into the region

Differential Induction



DI: Differential Induction

$$\frac{\Gamma, H \vdash F, \Delta \qquad \Gamma \vdash \forall^{\alpha}(H \rightarrow F'^{\theta_1,\ldots,\theta_n}_{x'_1,\ldots,x'_n}), \Delta}{\Gamma \vdash [x'_1 = \theta_1, \ldots, x'_n = \theta_n \& H]F, \Delta}$$

Notice: it is crucial that

$$D(F \vee G) = D(F) \wedge D(G)$$

or alternatively:

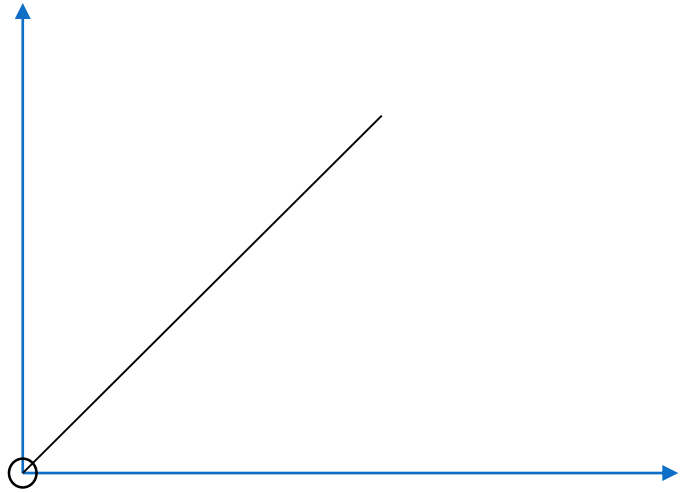
$$D(F \vee G) = (F \wedge D(F)) \vee (G \wedge D(G))$$

Differential Induction

UNSOUND!!! Differential Induction

$$\frac{\vdash \forall^{\alpha}(F \wedge H \to F'^{\theta_1,\ldots,\theta_n}_{x'_1,\ldots,x'_n})}{[H]F \vdash [x'_1 = \theta_1, \ldots, x'_n = \theta_n \& H]F}$$

$$\frac{\frac{*\,(unsound)}{\vdash \forall x.\,(x^2 \leq 0 \to 2x \leq 0)}}{x^2 \leq 0 \vdash [x' = 1]x^2 \leq 0}$$



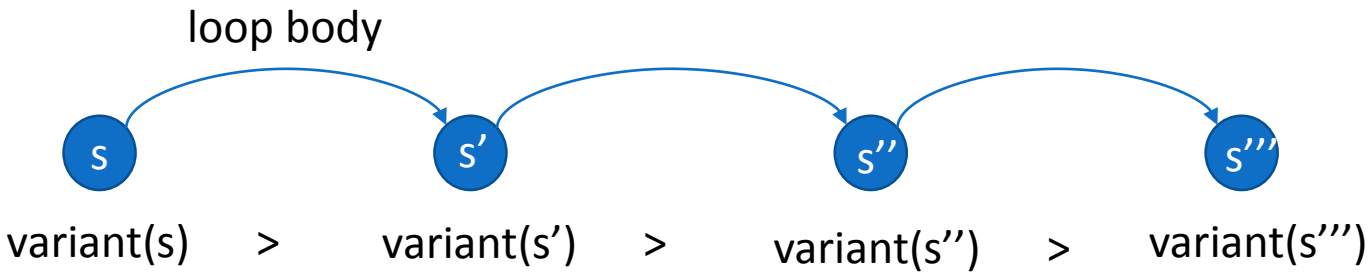
It is unsound to restrict the "induction step" to F only.

While Termination

Loop variant:
mathematical function on program state space whose value is monotonically decreasing wrt. a well-founded relation.

Well-founded relation:
Every nonempty subset $S \subseteq X$ of a class X has a minimal element:

$$\forall S \subseteq X. S \neq \emptyset \rightarrow \exists m \in S. \forall\, s \in S\ (s, m) \notin R$$



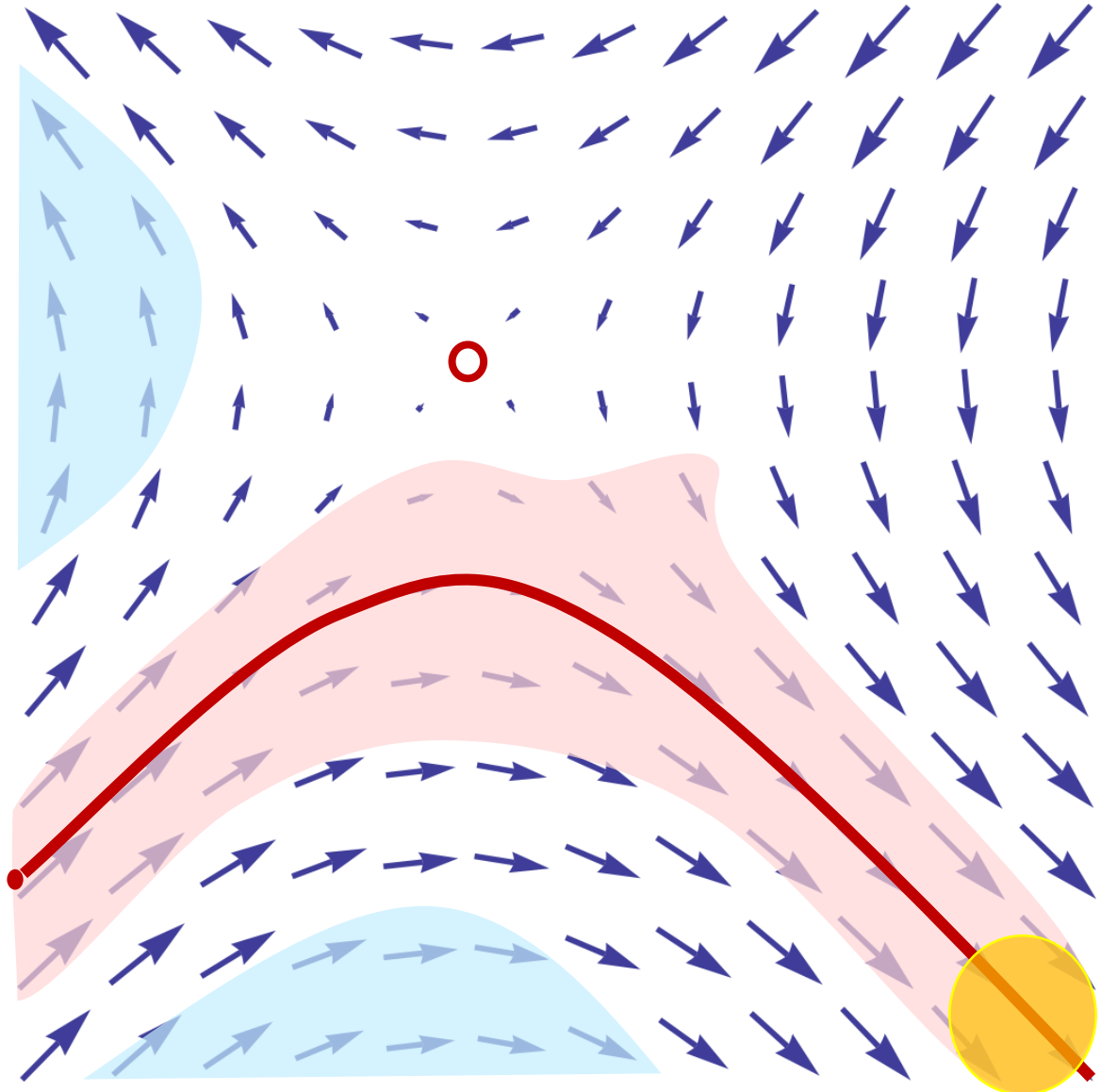
partial correctness:

$$\frac{\{I \wedge C\} S \{I\}}{\{I\}\ \textbf{while}\ C\ \textbf{do}\ S\ \{I\ \wedge \neg C\}}$$

total correctness:

$$\frac{< well\text{–}founded\ \ \{I \wedge C \wedge V = z\} S \{I \wedge V < z\}}{\{I\}\ \textbf{while}\ C\ \textbf{do}\ S\ \{I\ \wedge \neg C\}}$$

Differential Variant



DV: Differential Variant

$$\frac{\Gamma \vdash \exists \varepsilon > 0.\, \forall^{\alpha} (\neg F \wedge H \rightarrow (F' \geq \varepsilon)^{\theta_1,\ldots,\theta_n}_{x'_1,\ldots,x'_n}), \Delta}{\Gamma \vdash \langle x'_1 = \theta_1, \ldots, x'_n = \theta_n \& H \rangle F, \Delta}$$

where ~ is the weak negation, i.e., like ¬ except that $\sim(a \geq b) \equiv b \geq a$ and $\sim(\mathrm{a} > \mathrm{b}) \equiv a \leq b$

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