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A synoptic extension of the differential calculus derivation rules and beyond

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Abstract
The talk - part of a lecture course on Algorithm Engineering - presents an application-oriented attempt of a **synopsis** of the dc **product**, **quotient**, and **chain rule extensions**

based on

* the **normalized** differential operator notation

$$D^{(i)} = \frac{d^i}{i!dx^i} \; ; \; D^{(i)}f(x) = F^{(i)}$$

* a resulting two-dimensional formula representation by matrices and determinants **without numeric coefficients**

including

* Faa di Bruno’s formula,
* an extended Newton’s root finder,
* Taylor series representation of fractional functions,
* solution of linear differential equations, esp. applications in electrical engineering,
* a zoo of reciprocals of the factorial determinant patterns,
* sum of powers of natural numbers,
* polynomial handling, esp. a rational roots finder (wormholes through the irrationality).

A **prospective application area** in physics and engineering is the systematic design of static and dynamic, stationary and movable discrete structures (esp. metastructures with negative refractive index; 2D → 3D; macro, micro, nano range) which are able to guide a continuous wave or dynamic field around an object, to avoid considerable energetic interaction, impact absorption, destruction, interference, and observation.

Contents

1. Introduction

In the textbooks on differential calculus we can find three important derivative rules:

* **product rule**

$$(u(x)v(x))' = u(x)v'(x) + u'(x)v(x) \qquad (1)$$

and its extensions

* Leibniz identity

$$(u(x)v(x))^n = \sum_{i=0}^{n} \binom{n}{i} u^{(i)}(x) v^{(n-i)}(x) \qquad (2)$$

* $(u(x)v(x)w(x))' = u'(x)v(x)w(x) + u(x)v'(x)w(x) + u(x)v(x)w'(x)$ (3)

* **quotient rule**

* $$\left(\frac{u(x)}{v(x)}\right)' = \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2} \qquad (4.1)$$

* normalized

$$\frac{\left(\frac{u(x)}{v(x)}\right)'}{\left(\frac{1}{v(x)}\right)'} = u(x) - v(x)\frac{u'(x)}{v'(x)} \qquad (4.2)$$

* **chain rule**

$$(f(v(x)))' = (fov)' = f\,'(v(x))v'(x) \qquad (5.1)$$

and its extension

* Faa di Bruno’s formula, first derivatives

$$(f(v(x))' = f'v'$$
$$(f(v(x))'' = f'v'' + f''v'v'$$
$$(f(v(x))''' = f'v''' + 3f''v'v'' + f'''v'v'v'$$
$$(f(v(x))'''' = f'v'''' + 4f''v'v''' + 3f''v''v'' + 6f'''v'v'v'' + f''''v'v'v'v' \qquad (5.1\text{-}5)$$
$$f^{(i)} = f^{(i)}(v(x))\ ;\ v^{(i)} = v^{(i)}(x)$$

All this rules and their extensions are very important mathematical tools in physics and engineering .
In the following I want to present you an application-oriented attempt of a synopsis of these rules and extensions based on
* the **normalized** differential operator notation

$$D^{(i)} = \frac{d^i}{i!dx^i}\ ;\ D^{(i)}f(x) = F^{(i)}\ ;\ D^{(i)}u(x) = U^{(i)}\ ;\ D^{(i)}v(x) = V^{(i)} \qquad (6)$$

* a resulting two-dimensional formula representation by matrices and determinants **without numeric coefficients**.

2. From the normalized Leibniz identity to the extended quotient rule

With
$f(x) = u(x)v(x)\ ;\ F = UV$ (7.1)
and equ. (6) we obtain the normalized Leibniz identity in string representation

$$F^{(n)} = (UV)^{(n)} = \sum_{i=0}^{n} U^{(i)}V^{(n-i)} = \sum_{i=0}^{n} V^{(n-i)}U^{(i)}, \qquad (7.2)$$

and in matrix representation

$$\underbrace{\begin{bmatrix} F \\ F' \\ F'' \\ F''' \\ F^{(4)} \\ \vdots \end{bmatrix}}_{\vec{F}} = \begin{bmatrix} (UV) \\ (UV)' \\ (UV)'' \\ (UV)''' \\ (UV)^{(4)} \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} V & 0 & 0 & 0 & 0 & \\ V' & V & 0 & 0 & 0 & \\ V'' & V' & V & 0 & 0 & \\ V''' & V'' & V' & V & 0 & \\ V^{(4)} & V''' & V'' & V' & V & \\ & & \vdots & & & \ddots \end{bmatrix}}_{\underline{V}} \underbrace{\begin{bmatrix} U \\ U' \\ U'' \\ U''' \\ U^{(4)} \\ \vdots \end{bmatrix}}_{\vec{U}} \qquad (8)$$

The solution of equ. (8) for $\vec{U}$ and $U^{(4)}$, respectively, gives

$$\begin{bmatrix} U \\ U' \\ U'' \\ U''' \\ U^{(4)} \\ \vdots \end{bmatrix} = \begin{bmatrix} V & 0 & 0 & 0 & 0 & \\ V' & V & 0 & 0 & 0 & \\ V'' & V' & V & 0 & 0 & \\ V''' & V'' & V' & V & 0 & \\ V^{(4)} & V''' & V'' & V' & V & \\ & & \vdots & & & \ddots \end{bmatrix}^{-1} \begin{bmatrix} F \\ F' \\ F'' \\ F''' \\ F^{(4)} \\ \vdots \end{bmatrix} \qquad (9)$$

and via Cramer's rule for n=4

$$U^{(4)} = \frac{1}{V^{4+1}} \begin{vmatrix} V & 0 & 0 & 0 & F \\ V' & V & 0 & 0 & F' \\ V'' & V' & V & 0 & F'' \\ V''' & V'' & V' & V & F''' \\ V^{(4)} & V''' & V'' & V' & F^{(4)} \end{vmatrix}$$

$$= \frac{(-1)^4}{V^{4+1}} \begin{vmatrix} F & V & 0 & 0 & 0 \\ F' & V' & V & 0 & 0 \\ F'' & V'' & V' & V & 0 \\ F''' & V''' & V'' & V' & V \\ F^{(4)} & V^{(4)} & V''' & V'' & V' \end{vmatrix} \qquad (10)$$

Now we change in equ. (10) U↔ F to

$$F^{(4)} = \left(\frac{U}{V}\right)^{(4)} = \frac{(-1)^4}{V^{4+1}} \begin{vmatrix} U & V & 0 & 0 & 0 \\ U' & V' & V & 0 & 0 \\ U'' & V'' & V' & V & 0 \\ U''' & V''' & V'' & V' & V \\ U^{(4)} & V^{(4)} & V''' & V'' & V' \end{vmatrix} \qquad (11)$$

The heuristic extension 4 → n gives finally the normalized **extended quotient rule**

$$F^{(n)} = \left(\frac{U}{V}\right)^{(n)} = \frac{(-1)^n}{V^{n+1}} \underbrace{\begin{vmatrix} U & V & 0 & 0 & \cdots & 0 & 0 \\ U' & V' & V & 0 & \cdots & 0 & 0 \\ U'' & V'' & V' & V & & 0 & 0 \\ U''' & V''' & V'' & V' & & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \\ U^{(n-1)} & V^{(n-1)} & V^{(n-2)} & V^{(n-3)} & \cdots & V' & V \\ U^{(n)} & V^{(n)} & V^{(n-1)} & V^{(n-2)} & \cdots & V'' & V' \end{vmatrix}}_{(n+1)\times(n+1)\text{ determinant}} \qquad (12.1)$$

3. In the shadow of the determinant pattern of the extended quotient rule equ. (12.1)

$$\frac{d^n\left(\frac{u(x)}{v(x)}\right)}{n!dx^n} = D^{(n)}\left(\frac{u(x)}{v(x)}\right) = \left(\frac{U}{V}\right)^{(n)}$$

$$= \frac{(-1)^n}{V^{n+1}} \begin{vmatrix} U & V & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ U' & V' & V & 0 & 0 & 0 & 0 & 0 & 0 \\ U'' & V'' & V' & V & 0 & 0 & 0 & 0 & 0 \\ U''' & V''' & V'' & V' & V & 0 & 0 & 0 & 0 \\ U^{(4)} & V^{(4)} & V''' & V'' & V' & V & 0 & 0 & 0 \\ U^{(5)} & V^{(5)} & V^{(4)} & V''' & V'' & V' & V & 0 & 0 \\ & & & & & & V' & V & 0 \\ U^{(n-1)} & V^{(n-1)} & V^{(n-2)} & V^{(n-3)} & V^{(n-4)} & V^{(n-5)} & & V' & V \\ U^{(n)} & V^{(n)} & V^{(n-1)} & V^{(n-2)} & V^{(n-3)} & V^{(n-4)} & & V'' & V' \end{vmatrix}$$

$$= \frac{1}{(-V)^{n+1}} *$$

$$* \left(\begin{vmatrix} U' & V & 0 & 0 & 0 & 0 \\ U'' & V' & V & 0 & 0 & 0 \\ U''' & V'' & V' & V & 0 & 0 \\ U^{(4)} & V''' & V'' & V' & V & 0 \\ U^{(5)} & V^{(4)} & V''' & V'' & V' & V \\ U^{(n)} & V^{(n-1)} & & & & V' \end{vmatrix} V - U \begin{vmatrix} V' & V & 0 & 0 & 0 & 0 \\ V'' & V' & V & 0 & 0 & 0 \\ V''' & V'' & V' & V & 0 & 0 \\ V^{(4)} & V''' & V'' & V' & V & 0 \\ V^{(5)} & V^{(4)} & V''' & V'' & V' & V \\ V^{(n)} & V^{(n-1)} & & & & V' \end{vmatrix} \right)$$

$\det A_n$ nxn determinants $\det B_n$

(12.2)

This formulae cast a long constructivist shadow, including:

3.1 Faa di Bruno’s formula

$$\frac{d^n\left(f(v(x))\right)}{n!dx^n} = D^{(n)}\left(f(v(x))\right) = \begin{vmatrix} V' & V & 0 & 0 & 0 & 0 \\ V'' & V' & V & 0 & 0 & 0 \\ V''' & V'' & V' & V & 0 & 0 \\ V^{(4)} & V''' & V'' & V' & V & 0 \\ V^{(5)} & V^{(4)} & V''' & V'' & V' & V \\ V^{(n)} & V^{(n-1)} & & & & V' \end{vmatrix} = \det B_n$$

with the substitution

$$V^i \Rightarrow (-1)^i F^{(n-i)} = \left(\frac{d^{n-i} f(y)}{(n\text{-}i)!dy^{n-i}} |y=v(x) \right)$$

$i=0(1)n-1$

and

$$V^{(i)} = \frac{d^i v(x)}{i!dx^i} \ ;\ i=1(1)n$$

(14.1)

for n = 4

$$\frac{d^4\left(f(v(x))\right)}{4!dx^4} = D^{(4)}\left(f(v(x))\right) = \begin{vmatrix} V' & V & 0 & 0 \\ V'' & V' & V & 0 \\ V''' & V'' & V' & V \\ V^{(4)} & V''' & V'' & V' \end{vmatrix} = \det B_4$$

$$= V^0V'V'V'V' - 3V^1V'V'V'' + 2V^2V'V''' + V^2V''V'' - V^3V^{(4)}$$

with the substitution

$$V^i \Rightarrow (-1)^i F^{(n-i)} = \left(\frac{d^{n-i}f(y)}{(n-i)!dy^{n-i}}\bigg|_{y=v(x)}\right)$$

$$i=0(1)n-1$$

$$= F'V^{(4)} + 2F''V'V'' + F''V''V'' + 3F'''V'V'V'' + F^{(4)}V'V'V'V'$$

Open Problem: Interpret B_n for $V = \pm 1, \pm 2, \ldots$. (14.2,3)

http://mathworld.wolfram.com/FaadiBrunosFormula.html http://en.wikipedia.org/wiki/Fa%C3%A0_di_Bruno's_formula

Johnson, W.P. The Curious History of Faà di Bruno's Formula. American Mathematical Monthly **109** (2002), 217–234. http://www.maa.org/news/monthly217-234.pdf

3.2 Newton’s root finding algorithm and more

$$\frac{D^{(n)}\left(\frac{u(x)}{v(x)}\right)}{D^{(n)}\left(\frac{1}{v(x)}\right)} = \frac{\left(\frac{U}{V}\right)^{(n)}}{\left(\frac{1}{V}\right)^{(n)}} = U - V\frac{\begin{vmatrix} U' & V & 0 & 0 & 0 & 0 \\ U'' & V' & V & 0 & 0 & 0 \\ U''' & V'' & V' & V & 0 & 0 \\ U^{(4)} & V''' & V'' & V' & V & 0 \\ U^{(5)} & V^{(4)} & V''' & V'' & V' & V \\ U^{(n)} & V^{(n-1)} & & & & V' \end{vmatrix}}{\begin{vmatrix} V' & V & 0 & 0 & 0 & 0 \\ V'' & V' & V & 0 & 0 & 0 \\ V''' & V'' & V' & V & 0 & 0 \\ V^{(4)} & V''' & V'' & V' & V & 0 \\ V^{(5)} & V^{(4)} & V''' & V'' & V' & V \\ V^{(n)} & V^{(n-1)} & & & & V' \end{vmatrix}}$$

$$= U - V\frac{\det A_n}{\det B_n}$$

(15)

<u>If</u> $v(a) = 0$ <u>and</u> $u(a) \neq 0$ <u>then</u> $\lim_{x \to a} \frac{D^{(n)}\left(\frac{u(x)}{v(x)}\right)}{D^{(n)}\left(\frac{1}{v(x)}\right)} = u(x)$. (16)

Interpret the asignment

$$U := U - V\frac{\det A_n}{\det B_n} \qquad (17)$$

as body in a recursion loop.

For $u(x) = x$ we obtain the loop body of the extended Newton's root finder of $v(x)$ (18.1)

$$x := \frac{\left(\frac{x}{v(x)}\right)^{(n)}}{\left(\frac{1}{v(x)}\right)^{(n)}} = x - V\frac{\begin{vmatrix} V' & V & 0 & 0 & 0 \\ V'' & V' & V & 0 & 0 \\ V''' & V'' & V' & V & 0 \\ V^{(4)} & V''' & V'' & V' & V \\ V^{(n-1)} & V^{(n-2)} & & & V' \end{vmatrix}}{\begin{vmatrix} V' & V & 0 & 0 & 0 & 0 \\ V'' & V' & V & 0 & 0 & 0 \\ V''' & V'' & V' & V & 0 & 0 \\ V^{(4)} & V''' & V'' & V' & V & 0 \\ V^{(5)} & V^{(4)} & V''' & V'' & V' & V \\ V^{(n)} & V^{(n-1)} & & & & V' \end{vmatrix}} \qquad (18.2)$$

for n=1

$$x := \frac{\begin{vmatrix} x & V \\ 1 & V' \end{vmatrix}}{\begin{vmatrix} 1 & V \\ 0 & V' \end{vmatrix}} = x - \frac{V}{V'} = x - \frac{v(x)}{v'(x)};\ V^{(i)} = D^{(i)}v(x) \qquad (18.3)$$

n=2

$$x := \frac{\begin{vmatrix} x & V & 0 \\ 1 & V' & V \\ 0 & V'' & V' \end{vmatrix}}{\begin{vmatrix} 1 & V & 0 \\ 0 & V' & V \\ 0 & V'' & V' \end{vmatrix}} = x - \frac{VV'}{\begin{vmatrix} V' & V \\ V'' & V' \end{vmatrix}} = x - \frac{1}{\frac{V'}{V} - \frac{V''}{V'}} \qquad (18.4)$$

n=3

$$x := \frac{\begin{vmatrix} x & V & 0 & 0 \\ 1 & V' & V & 0 \\ 0 & V'' & V' & V \\ 0 & V''' & V'' & V' \end{vmatrix}}{\begin{vmatrix} 1 & V & 0 & 0 \\ 0 & V' & V & 0 \\ 0 & V'' & V' & V \\ 0 & V''' & V'' & V' \end{vmatrix}} = x - \frac{V\begin{vmatrix} V' & V \\ V'' & V' \end{vmatrix}}{\begin{vmatrix} V' & V & 0 \\ V'' & V' & V \\ V''' & V'' & V' \end{vmatrix}} \qquad (18.5)$$

Convergence!

<u>3.3 How about the special cases u(x) = v'(x) and v(x) = u'(x) ?</u>
<u>3.3.1 u(x) = v'(x)</u>

$$\frac{d^n\left(\frac{v'(x)}{v(x)}\right)}{n!dx^n} = D^{(n)}\left(\frac{v'(x)}{v(x)}\right) = \left(\frac{V'}{V}\right)^{(n)}$$

$$= \frac{(-1)^n}{V^{n+1}}\begin{vmatrix} 1V' & V & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2V'' & V' & V & 0 & 0 & 0 & 0 & 0 & 0 \\ 3V''' & V'' & V' & V & 0 & 0 & 0 & 0 & 0 \\ 4V^{(4)} & V''' & V'' & V' & V & 0 & 0 & 0 & 0 \\ 5V^{(5)} & V^{(4)} & V''' & V'' & V' & V & 0 & 0 & 0 \\ 6V^{(6)} & V^{(5)} & V^{(4)} & V''' & V'' & V' & V & 0 & 0 \\ & & & & & & V' & V & 0 \\ nV^{(n)} & V^{(n-1)} & V^{(n-2)} & V^{(n-3)} & V^{(n-4)} & V^{(n-5)} & & V' & V \\ (n+1)V^{(n+1)} & V^{(n)} & V^{(n-1)} & V^{(n-2)} & V^{(n-3)} & V^{(n-4)} & & V'' & V' \end{vmatrix}$$

Open problem: interpretation of this formula

<u>3.3.1 v(x) =u'(x)</u>

$$\frac{d^n\left(\dfrac{u(x)}{u'(x)}\right)}{n!dx^n} = D^{(n)}\left(\frac{u(x)}{u'(x)}\right) = \left(\frac{U}{U'}\right)^{(n)} = \frac{(-1)^n}{(U')^{n+1}} *$$

$$* \begin{vmatrix} U & 1U' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ U' & 2U'' & 1U' & 0 & 0 & 0 & 0 & 0 & 0 \\ U'' & 3U''' & 2U'' & 1U' & 0 & 0 & 0 & 0 & 0 \\ U''' & 4U^{(4)} & 3U''' & 2U'' & 1U' & 0 & 0 & 0 & 0 \\ U^{(4)} & 5U^{(5)} & 4U^{(4)} & 3U''' & 2U'' & 1U' & 0 & 0 & 0 \\ U^{(5)} & 6U^{(6)} & 5U^{(5)} & 4U^{(4)} & 3U''' & 2U'' & 1U' & 0 & 0 \\ & & & & & & 2U'' & 1U' & 0 \\ U^{(n-1)} & nU^{(n)} & (n-1)U^{(n-1)} & & & & & 2U'' & 1U' \\ U^{(n)} & (n+1)U^{(n+1)} & nU^{(n)} & & & & & 3U''' & 2U'' \end{vmatrix}$$

Open problem: interpretation of this formula

3.4 Taylor series representation of fractional functions

$$f(x) = \frac{u(x)}{v(x)}$$ **at a=0**

$$f(x) = \frac{u(x)}{v(x)} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} F^{(n)}(0) x^n \;\; ; \; F^{(n)}(0) \,|\, \text{equ. (12); x=0}$$

(21)

3.4.1 Special case: v(t) is normalized to v(0) = 1

For the special case v(0) = V(0) = 1 (22.1)

we obtain for f(x) the infinite determinant representations

$$f(x) = -\begin{vmatrix} 0 & x^0 & x^1 & x^2 & x^3 & x^4 & \rightarrow \\ U(0) & 1 & 0 & 0 & 0 & 0 & \\ U'(0) & V'(0) & 1 & 0 & 0 & 0 & \\ U''(0) & V''(0) & V'(0) & 1 & 0 & 0 & \\ U'''(0) & V'''(0) & V''(0) & V'(0) & 1 & 0 & \\ U^{(4)}(0) & V^{(4)}(0) & V'''(0) & V''(0) & V'(0) & 1 & \\ \downarrow & \ddots & \mathrm{K} & & & & \end{vmatrix}$$

$$= -\begin{vmatrix} 0 & U(0) & U'(0) & U''(0) & U'''(0) & U^{(4)}(0) & \rightarrow \\ x^0 & 1 & V'(0) & V''(0) & V'''(0) & V^{(4)}(0) & \\ x^1 & 0 & 1 & V'(0) & V''(0) & V'''(0) & \\ x^2 & 0 & 0 & 1 & V'(0) & V''(0) & \\ x^3 & 0 & 0 & 0 & 1 & V'(0) & \\ x^4 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & \ddots & \mathrm{K} & & & & \end{vmatrix} \qquad (22.2)$$

3.4.2

$$u(x)=\sin x;\ v(x)=\cos x;\ f(x)=\tan x=\frac{\sin x}{\cos x} \qquad (23.1)$$

$$u(0)=0;\ u'(0)=1;\ u''(0)=1;\ u'''(0)=-1;\ u^{(4)}(0)=0 \ \rightarrow \text{per.4} \qquad (23.2)$$

$$v(0)=1;\ v'(0)=0;\ v''(0)=-1;\ v'''(0)=0;\ v^{(4)}(0)=1 \ \rightarrow \text{per.4} \qquad (23.3)$$

$$U^{(i)}(0) = \frac{u^{(i)}(0)}{i!};\ \ V^{(i)}(0) = \frac{v^{(i)}(0)}{i!} \qquad (23.4)$$

$$F^{(n)}(0)=\frac{(-1)^n}{1}\underbrace{\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \frac{1}{1!} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & -\frac{1}{2!} & 0 & 1 & 0 & 0 & 0 & 0 & \\ -\frac{1}{3!} & 0 & -\frac{1}{2!} & 0 & 1 & 0 & 0 & 0 & \\ 0 & \frac{1}{4!} & 0 & -\frac{1}{2!} & 0 & 1 & 0 & 0 & \\ \frac{1}{5!} & 0 & \frac{1}{4!} & 0 & -\frac{1}{2!} & 0 & 1 & 0 & \\ 0 & -\frac{1}{6!} & 0 & \frac{1}{4!} & 0 & -\frac{1}{2!} & 0 & 1 & \\ -\frac{1}{7!} & 0 & -\frac{1}{6!} & 0 & \frac{1}{4!} & 0 & -\frac{1}{2!} & 0 & \\ & & & & \vdots & & & & \ddots \end{vmatrix}}_{(n+1)\times(n+1)\text{ determinant}} \tag{23.5}$$

From equ. (23.5) follows

$$F^{(0)}(0)=F^{(2)}(0)=F^{(4)}(0)=F^{(6)}(0)=\cdots=F^{(2k)}(0)=0 \tag{23.6}$$

$$F^{(1)}(0)=+\begin{vmatrix}\frac{1}{1!}\end{vmatrix}=1 \tag{23.8}$$

$$F^{(3)}(0)=-\begin{vmatrix}\frac{1}{1!} & 1 \\ -\frac{1}{3!} & -\frac{1}{2!}\end{vmatrix}=\frac{1}{3} \tag{23.9}$$

$$F^{(5)}(0) = +\begin{vmatrix} \frac{1}{1!} & 1 & 0 \\ -\frac{1}{3!} & -\frac{1}{2!} & 1 \\ \frac{1}{5!} & \frac{1}{4!} & -\frac{1}{2!} \end{vmatrix} = \frac{2}{15} \qquad (23.10)$$

$$F^{(7)}(0) = -\begin{vmatrix} \frac{1}{1!} & 1 & 0 & 0 \\ -\frac{1}{3!} & -\frac{1}{2!} & 1 & 0 \\ \frac{1}{5!} & \frac{1}{4!} & -\frac{1}{2!} & 1 \\ -\frac{1}{7!} & -\frac{1}{6!} & \frac{1}{4!} & -\frac{1}{2!} \end{vmatrix} = \frac{17}{315} \qquad (23.11)$$

So we obtain finally

$$\tan x = \begin{vmatrix} \frac{1}{1!} \end{vmatrix} x + \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} \\ \frac{1}{3!} & \frac{1}{2!} \end{vmatrix} x^3 + \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix} x^5 + \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{0!} & 0 \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix} x^7 + \ldots$$

$$= \begin{vmatrix} 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \rightarrow \\ -x^1 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ +x^3 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ -x^5 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ +x^7 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ -x^9 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow \ \ddot{} & K & & & & & \end{vmatrix}$$

$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \ldots \qquad (24)$$

3.4.2

u(x)=1, v(x)=cos x; f(x)= $\frac{1}{\cos x}$

$$\frac{1}{\cos x} = 1 + \begin{vmatrix}\frac{1}{2!}\end{vmatrix} x^2 + \begin{vmatrix}\frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{4!} & \frac{1}{2!}\end{vmatrix} x^4 + \begin{vmatrix}\frac{1}{2!} & \frac{1}{0!} & 0 \\ \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!}\end{vmatrix} x^6 + \begin{vmatrix}\frac{1}{2!} & \frac{1}{0!} & 0 & 0 \\ \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!}\end{vmatrix} x^8 + \ldots$$

$$= \begin{vmatrix} +x^0 & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \rightarrow \\ -x^2 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ +x^4 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ -x^6 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ +x^8 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ -x^{10} & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & \ddots & & & & & \end{vmatrix}$$

$$= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{1385}{40320}x^8 + \ldots$$

(25)

Extensions

$$\frac{1}{\cos^2 x} = 1 + \begin{vmatrix}\frac{2}{2!}\end{vmatrix} x^2 + \begin{vmatrix}\frac{2}{2!} & \frac{1}{0!} \\ \frac{8}{4!} & \frac{2}{2!}\end{vmatrix} x^4 + \begin{vmatrix}\frac{2}{2!} & \frac{1}{0!} & 0 \\ \frac{8}{4!} & \frac{2}{2!} & \frac{1}{0!} \\ \frac{32}{6!} & \frac{8}{4!} & \frac{2}{2!}\end{vmatrix} x^6 + \begin{vmatrix}\frac{2}{2!} & \frac{1}{0!} & 0 & 0 \\ \frac{8}{4!} & \frac{2}{2!} & \frac{1}{0!} & 0 \\ \frac{32}{6!} & \frac{8}{4!} & \frac{2}{2!} & \frac{1}{0!} \\ \frac{128}{8!} & \frac{32}{6!} & \frac{8}{4!} & \frac{2}{2!}\end{vmatrix} x^8 + \ldots$$

$$= 1 + x^2 + \frac{2}{3}x^4 + \frac{17}{45}x^6 + \frac{62}{315}x^8 + \ldots$$

(26)

$$\frac{1}{\cos^3 x} = 1 + \begin{vmatrix} A_2 \end{vmatrix} x^2 + \begin{vmatrix} A_2 & 1 \\ A_4 & A_2 \end{vmatrix} x^4 + \begin{vmatrix} A_2 & 1 & 0 \\ A_4 & A_2 & 1 \\ A_6 & A_4 & A_2 \end{vmatrix} x^6 + \begin{vmatrix} A_2 & 1 & 0 & 0 \\ A_4 & A_4 & 1 & 0 \\ A_6 & A_2 & A_2 & 1 \\ A_8 & A_6 & A_4 & A_2 \end{vmatrix} x^8 + \ldots$$

$$A_{2k} = \frac{\frac{3}{4}(3^{2k-1} + 1)}{(2k)!} \ ; \ k=1,2,3,4,....$$

$$A_2 = \frac{3}{2!} \ ; \ A_4 = \frac{21}{4!} \ ; \ A_6 = \frac{183}{6!} \ ; \ A_8 = \frac{1641}{8!} \ ; \ ....$$

3.4.3 (27)

$$u(x)=1,\ v(x)=\frac{\sin x}{x}\ ;\ f(x)=\frac{1}{\frac{\sin x}{x}}$$

$$\frac{x}{\sin x}=1+\begin{vmatrix}\frac{1}{3!}\end{vmatrix}x^2+\begin{vmatrix}\frac{1}{3!} & \frac{1}{1!}\\ \frac{1}{5!} & \frac{1}{3!}\end{vmatrix}x^4+\begin{vmatrix}\frac{1}{3!} & \frac{1}{1!} & 0\\ \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!}\\ \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!}\end{vmatrix}x^6+\begin{vmatrix}\frac{1}{3!} & \frac{1}{1!} & 0 & 0\\ \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & 0\\ \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!}\\ \frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!}\end{vmatrix}x^8+\ldots$$

$$=\begin{vmatrix}+x^0 & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \rightarrow\\ -x^2 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ +x^4 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ -x^6 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ +x^8 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ -x^{10} & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & \ddots & \mathrm{K} & & & & \end{vmatrix}$$

$$=1+\frac{1}{6}x^2+\frac{7}{360}x^4+\frac{31}{15120}x^6+\frac{127}{604800}x^8+\ldots \qquad (28)$$

3.4.4 Gudermannian

Gudermannian function

$gd(x) = \tan^{-1}(\sinh(x))$

$$= x - \begin{vmatrix} \frac{1}{2!} \end{vmatrix} \frac{x^3}{3} + \begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{4!} & \frac{1}{2!} \end{vmatrix} \frac{x^5}{5} - \begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} & 0 \\ \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix} \frac{x^7}{7} + \begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} & 0 & 0 \\ \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix} \frac{x^9}{9} \mp \dots$$

$$= x - \frac{1}{6}x^3 + \frac{1}{24}x^5 - \frac{61}{5040}x^7 + \frac{277}{72576}x^9 \mp \dots$$

Inverse Gudermannian function

$$gd^{-1}(x) = \ln(\tan x + \frac{1}{\cos x}) = \frac{gd(\sqrt{-1}\ x)}{\sqrt{-1}}$$

$$= x + \begin{vmatrix} \frac{1}{2!} \end{vmatrix} \frac{x^3}{3} + \begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{4!} & \frac{1}{2!} \end{vmatrix} \frac{x^5}{5} + \begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} & 0 \\ \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix} \frac{x^7}{7} + \begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} & 0 & 0 \\ \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix} \frac{x^9}{9} + \dots$$

$$= x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{61}{5040}x^7 + \frac{277}{72576}x^9 + \dots$$

(29.1.2)

$$\left.\begin{matrix} gd(x) \\ gd^{-1}(x) \end{matrix}\right\} = \begin{vmatrix} \frac{x^1}{1} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \rightarrow \\ \pm\frac{x^3}{3} & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ \frac{x^5}{5} & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ \pm\frac{x^7}{7} & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ \frac{x^9}{9} & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ \pm\frac{x^{11}}{11} & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow \ddot{} & \text{K} & & & & & \end{vmatrix}$$

(29.3)

3.4.5

$$e^x = 1 + \begin{vmatrix} \frac{1}{1!} \end{vmatrix} x + \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} \\ \frac{1}{2!} & \frac{1}{1!} \end{vmatrix} x^2 + \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} \end{vmatrix} x^3 + \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 & 0 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} \end{vmatrix} x^4 + \ldots$$

$$= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots$$

.

$$
= \begin{vmatrix}
+x^0 & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \rightarrow \\
-x^1 & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \\
+x^2 & 0 & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \\
-x^3 & 0 & 0 & \frac{1}{0!} & \frac{1}{1!} & \\
+x^4 & 0 & 0 & 0 & \frac{1}{0!} & \\
\downarrow & & & & & ..
\end{vmatrix}
\qquad (30.1,2)
$$

3.4.6

$$
\frac{x}{\ln(1+x)} = 1 + \begin{vmatrix} \frac{1}{2} \end{vmatrix} x + \begin{vmatrix} \frac{1}{2} & \frac{1}{1} \\ \frac{1}{3} & \frac{1}{2} \end{vmatrix} x^2 + \begin{vmatrix} \frac{1}{2} & \frac{1}{1} & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{1} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \end{vmatrix} x^3 + \begin{vmatrix} \frac{1}{2} & \frac{1}{1} & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{1} & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & \frac{1}{1} \\ \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \end{vmatrix} x^4 + \ldots
$$

$$
= 1 + \frac{1}{2}x - \frac{1}{12}x^2 + \frac{1}{24}x^3 - \frac{19}{720}x^4 + \frac{3}{160}x^5 - \frac{863}{60480}x^6 + \frac{1375}{120960}x^7
$$

$$
+ \frac{275}{24192}x^8 - \frac{33953}{3628800}x^9 + \frac{8183}{1036800}x^{10} - \frac{3250433}{479001600}x^{11} + \frac{4671}{788480}x^{12} - \ldots
\qquad (31)
$$

3.4.7

$$
u(x) = \sum_{i=0}^{\infty} a_i x^i; \ \mathrm{v}(x) = \sum_{i=0}^{\infty} b_i x^i \qquad (32.1)
$$

$$
U^{(i)}(0) = a_i; \ V^{(i)}(0) = b_i \qquad (32.2)
$$

$$F^{(n)}(0) = \frac{(-1)^n}{b_0^{n+1}} \begin{vmatrix} a_0 & b_0 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & b_1 & b_0 & 0 & \cdots & 0 & 0 \\ a_2 & b_2 & b_1 & b_0 & & 0 & 0 \\ a_3 & b_3 & b_2 & b_1 & \ddots & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\ a_{n-1} & b_{n-1} & b_{n-2} & b_{n-3} & \cdots & b_1 & b_0 \\ a_n & b_n & b_{n-1} & b_{n-2} & \cdots & b_2 & b_1 \end{vmatrix} \tag{32.3}$$

$$F(0) = \frac{a_0}{b_0} \tag{32.4}$$

$$F'(0) = \frac{-1}{b_0^2} \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} \tag{32.5}$$

$$F''(0) = \frac{1}{b_0^3} \begin{vmatrix} a_0 & b_0 & 0 \\ a_1 & b_1 & b_0 \\ a_2 & b_2 & b_1 \end{vmatrix} \tag{32.6}$$

$$\frac{\sum_{i=0}^{\infty} a_i x^i}{\sum_{i=0}^{\infty} b_i x^i} = \sum_{n=0}^{\infty} F^{(n)}(0) x^n \tag{32.7}$$

$$= -\begin{vmatrix} 0 & a_0 & a_1 & a_2 & a_3 & \rightarrow \\ x^0 & 1 & b_1 & b_2 & b_3 & \\ x^1 & 0 & 1 & b_1 & b_2 & \\ x^2 & 0 & 0 & 1 & b_1 & \\ x^3 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & .. \end{vmatrix} \quad \text{with } b_0 = 1 \tag{32.8}$$

Inversion for $a_0 = 1$; $(\forall i)\ a_n = 0,\ n = 1(1)\infty$

$$b_0 = \frac{1}{F^{(0)}(0)}$$

$$b_n = \frac{(-1)^n}{(F^{(0)}(0))^{n+1}} \begin{vmatrix} 1 & F^{(0)}(0) & 0 & \cdots & 0 \\ 0 & F^{(1)}(0) & F^{(0)}(0) & \cdots & 0 \\ 0 & F^{(2)}(0) & F^{(1)}(0) & \cdots & 0 \\ 0 & F^{(3)}(0) & F^{(2)}(0) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & F^{(n-1)}(0) & F^{(n-2)}(0) & \cdots & F^{(0)}(0) \\ 0 & F^{(n)}(0) & F^{(n-1)}(0) & \cdots & F^{(1)}(0) \end{vmatrix} \quad (33)$$

3.4.8

$$u(x) = (1+x)^k - (1-x)^k;\ \mathrm{v}(x) = (1+x)^k + (1-x)^k \quad (34.1)$$

With

$$U^{(i)}(0) = \frac{1}{i!}u^{(i)}(0);\ V^{(i)}(0) = \frac{1}{i!}v^{(i)}(0) \quad (34.2)$$

we have

$$U^{(2l)}(0) = 0,\ U^{(2l+1)}(0) = 2\binom{k}{2l+1};\ l = 0,1,2,\cdots \quad (34.3)$$

$$V^{(2l)}(0) = 2\binom{k}{2l},\ \mathrm{V}^{(2l+1)}(0) = 0;\ l = 0,1,2,\cdots \quad (34.4)$$

and

$$F^{(n)}(0) = \underbrace{\begin{vmatrix} 0 & \binom{k}{0} & 0 & 0 & 0 & 0 & 0 & 0 & \\ \binom{k}{1} & 0 & \binom{k}{0} & 0 & 0 & 0 & 0 & 0 & \\ 0 & \binom{k}{2} & 0 & \binom{k}{0} & 0 & 0 & 0 & 0 & \\ \binom{k}{3} & 0 & \binom{k}{2} & 0 & \binom{k}{0} & 0 & 0 & 0 & \\ 0 & \binom{k}{4} & 0 & \binom{k}{2} & 0 & \binom{k}{0} & 0 & 0 & \\ \binom{k}{5} & 0 & \binom{k}{4} & 0 & \binom{k}{2} & 0 & \binom{k}{0} & 0 & \\ 0 & \binom{k}{6} & 0 & \binom{k}{4} & 0 & \binom{k}{2} & 0 & \binom{k}{0} & \\ \binom{k}{7} & 0 & \binom{k}{6} & 0 & \binom{k}{4} & 0 & \binom{k}{2} & 0 & \\ & & & & \vdots & & & & \ddots \end{vmatrix}}_{\text{(n+1)×(n+1) determinant}} \tag{34.5}$$

With

$$F^{(2l)}(0) = 0 \tag{34.6}$$

$$\begin{vmatrix} \binom{k}{1} & \binom{k}{0} & 0 & & 0 \\ \binom{k}{3} & \binom{k}{2} & \binom{k}{0} & & 0 \end{vmatrix}$$

$$(l+1)\times(l+1) \text{ determinant} \tag{34.7}$$

we obtain finally

$$f(x)=\frac{(1+x)^k-(1-x)^k}{(1+x)^k+(1-x)^k}=\binom{k}{1}x-\begin{vmatrix}\binom{k}{1} & \binom{k}{0}\\ \binom{k}{3} & \binom{k}{2}\end{vmatrix}x^3+\begin{vmatrix}\binom{k}{1} & \binom{k}{0} & 0\\ \binom{k}{3} & \binom{k}{2} & \binom{k}{0}\\ \binom{k}{5} & \binom{k}{4} & \binom{k}{2}\end{vmatrix}x^5$$

$$-\begin{vmatrix}\binom{k}{1} & \binom{k}{0} & 0 & 0\\ \binom{k}{3} & \binom{k}{2} & \binom{k}{0} & 0\\ \binom{k}{5} & \binom{k}{4} & \binom{k}{2} & \binom{k}{0}\\ \binom{k}{7} & \binom{k}{6} & \binom{k}{4} & \binom{k}{2}\end{vmatrix}x^7\pm\cdots \tag{35.1}$$

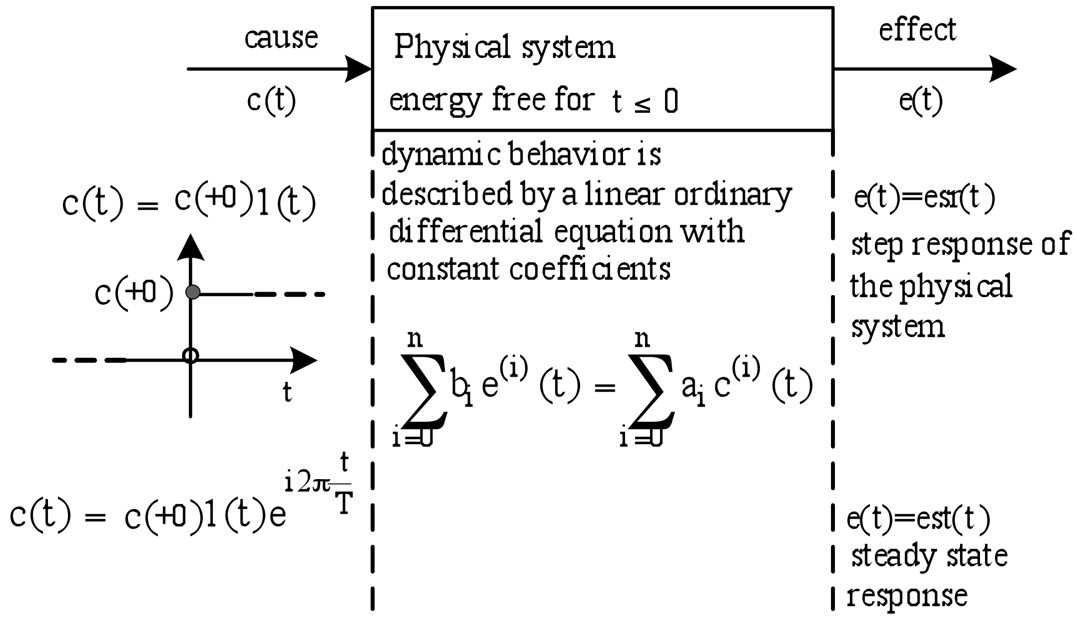
$$= -\begin{vmatrix} 0 & \binom{k}{1} & \binom{k}{3} & \binom{k}{5} & \binom{k}{7} & \rightarrow \\ x^1 & \binom{k}{0} & \binom{k}{2} & \binom{k}{4} & \binom{k}{6} & \\ x^3 & 0 & \binom{k}{0} & \binom{k}{2} & \binom{k}{4} & \\ x^5 & 0 & 0 & \binom{k}{0} & \binom{k}{2} & \\ x^7 & 0 & 0 & 0 & \binom{k}{0} & \\ \downarrow & & & & & .. \end{vmatrix} \tag{35.2}$$

Note

$$\binom{k}{i} = \left\langle \begin{matrix} \dfrac{k!}{i!(k-i)!} \\ \\ 0 \end{matrix} \right. \text{ iff } \begin{matrix} k \ge i \\ \\ k < i \end{matrix} \quad ; k, i \in \mathbb{N} \tag{36}$$

<u>4. Solution of linear differential equations, esp. applications in electrical engineering</u>

Given



(37)

* Step response esr(t) and
* steady state response est(t) on a complex exponential (rotation operator); frequency response

are the most important dynamic input-output relations for the evaluation of such a system.

Find

Taylor series representation of esr(t) via Laplace transform

Solution

$$\frac{esr(t)}{c(+0)} = \sum_{k=0}^{\infty} \frac{esr^{(k)}(+0)}{c(+0)} \frac{t^k}{k!} \ ; t \geq 0$$

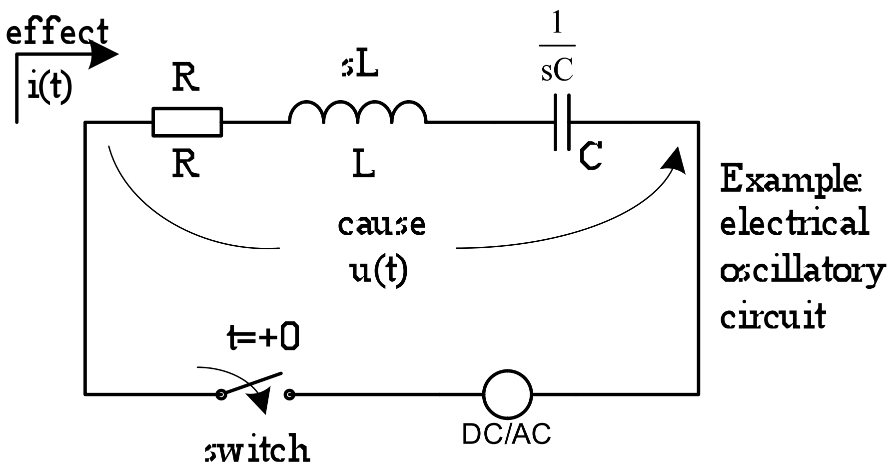
$$\frac{esr^{(k)}(+0)}{c(+0)} = \frac{(-1)^k}{b_n^{\ k+1}} \begin{vmatrix} a_n & b_n & 0 & 0 & 0 & & 0 \\ a_{n-1} & b_{n-1} & 0 & 0 & 0 & & \\ a_{n-2} & b_{n-1} & b_{n-1} & 0 & 0 & & \\ a_{n-3} & b_{n-1} & b_{n-1} & b_{n-1} & 0 & & \\ a_{n-4} & b_{n-1} & b_{n-1} & b_{n-1} & b_{n-1} & & \\ & & & & & & \\ a_{n-k} & b_{n-1} & b_{n-1} & & & & b_{n-1} \end{vmatrix}$$

(k+1)x(k+1) determinant

(38.1,2)

$$\left.\frac{esr(t)}{c(+0)}\right| b_n = 1 = - \begin{vmatrix} 0 & a_n & a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} & \rightarrow \\ \frac{t^0}{0!} & 1 & b_{n-1} & b_{n-2} & b_{n-3} & b_{n-4} & \\ \frac{t^1}{1!} & 0 & 1 & b_{n-1} & b_{n-2} & b_{n-3} & \\ \frac{t^2}{2!} & 0 & 0 & 1 & b_{n-1} & b_{n2} & \\ \frac{t^3}{3!} & 0 & 0 & 0 & 1 & b_{n-1} & \\ \frac{t^4}{4!} & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & .. \end{vmatrix}$$

(38.3)



energy free for t≤0

$$\rightarrow i_L(+0) = i(+0) = 0$$

$$u_C(+0) = 0$$

$$i''(t) + \frac{R}{L} i'(t) + \frac{1}{LC} i(t) = \frac{1}{L} u'(t) \quad ; \ \mathrm{L,C} \neq 0$$

$$\text{n=2; } a_2 = 0; \ a_1 = \frac{1}{L}; \ a_0 = 0$$

$$b_2 = 1; \ b_1 = \frac{R}{L}; \ b_0 = \frac{1}{LC} \qquad (39)$$

“Screenplay” solution

$$isr(t) = u(t)\left(\frac{1}{L} t - \frac{R}{L^2} \frac{t^2}{2!} + \frac{1}{L} \begin{vmatrix} \frac{R}{L} & 1 \\ \frac{1}{LC} & \frac{R}{L} \end{vmatrix} \frac{t^3}{3!} - \frac{1}{L} \begin{vmatrix} \frac{R}{L} & 1 & 0 \\ \frac{1}{LC} & \frac{R}{L} & 1 \\ 0 & \frac{1}{LC} & \frac{R}{L} \end{vmatrix} \frac{t^4}{4!} \pm \cdots \right)$$

(40)

Normalized solution

$$isr(t)=\frac{u(t)}{R}*$$

$$*\left(\frac{Rt}{L}-\left(\frac{Rt}{L}\right)^{2}\frac{1}{2!}+\begin{vmatrix}1 & 1\\ \frac{L}{R^{2}C} & 1\end{vmatrix}\left(\frac{Rt}{L}\right)^{3}\frac{1}{3!}-\begin{vmatrix}1 & 1 & 0\\ \frac{L}{R^{2}C} & 1 & 1\\ 0 & \frac{L}{R^{2}C} & 1\end{vmatrix}\left(\frac{Rt}{L}\right)^{4}\frac{1}{4!}\pm\cdots\right)$$

(41)

Normalized branched continued fraction representation

$$isr(t)=\frac{u(t)}{R}\left(\frac{x}{1}+\frac{\frac{x}{2}}{1-\frac{x}{2}}+\frac{\frac{x}{3}\left(1-\frac{\delta}{1}\right)}{1-\frac{x}{3}\left(1-\frac{\delta}{1}\right)}+\frac{\frac{x}{4}\left(1-\cfrac{\delta}{1-\cfrac{\delta}{1}}\right)}{1-\frac{x}{4}\left(1-\cfrac{\delta}{1-\cfrac{\delta}{1}}\right)}+\frac{\frac{x}{5}\left(1-\cfrac{\delta}{1-\cfrac{\delta}{1-\cfrac{\delta}{1}}}\right)}{1-\frac{x}{5}\left(1-\cfrac{\delta}{1-\cfrac{\delta}{1-\cfrac{\delta}{1}}}\right)}+\cdots\right)$$

$$x=\frac{R}{L}t,\ \delta=\frac{L}{R^{2}C}\neq 1$$

(42)

5. A zoo of reciprocals of the factorial determinant patterns

5.1 Connections with Bernoulli and Euler numbers

$$B_{2n} = \frac{(-1)^{n-1}(2n)!}{2^{2n}(2^{2n}-1)} \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & 0 & \rightarrow \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{0!} & 0 & 0 & \rightarrow \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 & \rightarrow \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & \rightarrow \\ \frac{1}{9!} & \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \rightarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow &] \end{vmatrix} \quad ; \; n=1,2,3,4,..$$

determinant of nth order

$$B_{2n} = \frac{(-1)^{n-1}(2n)!}{2^{2n}-2} \begin{vmatrix} \frac{1}{3!} & \frac{1}{1!} & 0 & 0 & 0 & \rightarrow \\ \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & 0 & 0 & \rightarrow \\ \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & 0 & \rightarrow \\ \frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & \rightarrow \\ \frac{1}{11!} & \frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \rightarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow &] \end{vmatrix} \quad ; \; n=1,2,3,4,..$$

determinant of nth order

(43.1,2)

$$E_{2n} = (-1)^n (2n)! \begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} & 0 & 0 & 0 & \rightarrow \\ \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 & 0 & \rightarrow \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 & \rightarrow \\ \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & \rightarrow \\ \frac{1}{10!} & \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \rightarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow &] \end{vmatrix} ; \; n=1,2,3,4,...$$

determinant of nth order

(44.1)

$B_0 = 1;\ B_1 = -\frac{1}{2};\ B_2 = \frac{1}{6};\ B_4 = -\frac{1}{30};\ B_6 = \frac{1}{42};\ B_8 = -\frac{1}{30};\ ...$

$B_{2k+1} = 0;\ k = 1,2,3,...$

http://mathworld.wolfram.com/BernoulliNumber.html (43.3)

$E_2 = -1;\ E_4 = 5;\ E_6 = -61;\ E_8 = 1385;\ E_{10} = -50521;\ ...$

$E_{2k+1} = 0;\ k = 0,1,2,3,..$

http://mathworld.wolfram.com/EulerNumber.html (44.2)

$$\begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & 0 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{0!} & 0 & 0 & 0 \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 & 0 \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 \\ \frac{1}{9!} & \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{11!} & \frac{1}{10!} & \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix} = \frac{2^{2n}(2^{2n}-1)}{2^{2n}-2} \begin{vmatrix} \frac{1}{3!} & \frac{1}{1!} & 0 & 0 & 0 & 0 \\ \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & 0 & 0 & 0 \\ \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & 0 & 0 \\ \frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & 0 \\ \frac{1}{11!} & \frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} \\ \frac{1}{13!} & \frac{1}{11!} & \frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} \end{vmatrix} \quad (45)$$

5.2 A connection with p

The sequence

$$g(n) = \frac{\begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} & 0 & 0 & \rightarrow \\ \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 & \rightarrow \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & \rightarrow \\ \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \rightarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \searrow \end{vmatrix} \text{ determinant nth order}}{\begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & 0 & \rightarrow \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{0!} & 0 & 0 & \rightarrow \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 & \rightarrow \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & \rightarrow \\ \frac{1}{9!} & \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \rightarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \searrow \end{vmatrix} \text{ determinant (n+1) order}} \quad ; \quad n=1,2,3,4,\ldots \tag{46.1}$$

converges linearly with the difference quotient

$$\lim_{n\to\infty} \frac{g(n)-g(n+1)}{g(n+1)-g(n+2)} = 9$$

to

$$\lim_{n\to\infty} g(n) = \frac{\pi}{2} .$$

(46.2,3)

5.3 A connection with e

$$e = \begin{vmatrix} +1 & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \rightarrow \\ -1 & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \\ +1 & 0 & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \\ -1 & 0 & 0 & \frac{1}{0!} & \frac{1}{1!} & \\ +1 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & .. \end{vmatrix}$$

$$\frac{1}{e} = \begin{vmatrix} 1 & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \rightarrow \\ 1 & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \\ 1 & 0 & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \\ 1 & 0 & 0 & \frac{1}{0!} & \frac{1}{1!} & \\ 1 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & .. \end{vmatrix}$$

(47.1,3)

5.4 Connections with other mathematical constants

5.5 Connections with prines

5.6 Miscellaneous

$$\frac{1}{\frac{1}{n!}}\begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & 0 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 & 0 \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 \\ \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{(n-2)!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} \end{vmatrix} = 1 \quad ; n = 1,2,3,..$$

nxn determinant

(48)

$$\frac{1}{\frac{1}{n!}}\begin{vmatrix} \frac{1}{2!} & \frac{1}{1!} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 & 0 & 0 & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 & 0 & 0 \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 & 0 \\ \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix} = B_n \quad ; n = 2,3,4,$$

nxn determinant

(49)

$$\frac{1}{\frac{1}{n!}}\begin{vmatrix} \frac{1}{3!} & \frac{1}{2!} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 0 & 0 & 0 & 0 \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 0 & 0 & 0 \\ \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 0 & 0 \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 0 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} \\ \frac{1}{(n+2)!} & \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} \end{vmatrix} = ???(n) \quad ; n = 1,2.3,..$$

nxn determinant

(50)

6. Sum of powers of natural numbers

6.1 Sum

$$s(n,m) = \sum_{i=1}^{n} i^m \; ; \quad s(n,0) = n \; ; \; s(n,1) = \frac{1}{2}n(n+1) \; ;$$

$$s(n,2) = \frac{1}{6}n(n+1)(2n+1) \; ;$$

$$s(n,3) = (\frac{1}{2}n(n+1))^2 = (s(n,1))^2$$

$$\frac{s(n,m)}{s(n,0)} =$$

$$\frac{1}{(m+1)!}\begin{vmatrix} (-n)^0 & \binom{1}{0} & 0 & 0 & 0 & 0 & 0 \\ (-n)^1 & \binom{2}{0} & \binom{2}{1} & 0 & 0 & 0 & 0 \\ (-n)^2 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & 0 & 0 & 0 \\ (-n)^3 & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & & 0 & 0 \\ & & & & & & 0 \\ (-n)^{m-1} & \binom{m}{0} & \binom{m}{1} & \binom{m}{2} & & \binom{m}{m-2} & \binom{m}{m-1} \\ (-n)^m & \binom{m+1}{0} & \binom{m+1}{1} & \binom{m+1}{2} & & \binom{m+1}{m-2} & \binom{m+1}{m-1} \end{vmatrix}$$

(m+1)x(m+1) determinant

(51)

6.1.1 Sum of even powers

$$\frac{f(n,2k)}{f(n,2)} = \frac{\sum_{i=1}^{n} i^{2k}}{\sum_{i=1}^{n} i^{2}} = \frac{(-1)^{k+1}}{\prod_{i=2}^{k}(2i+1)} *$$

$$* \begin{vmatrix}
p^0 & \binom{1}{0}+\binom{2}{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
p^1 & \binom{3}{0} & \binom{2}{1}+\binom{3}{2} & 0 & 0 & 0 & 0 & 0 \\
p^2 & 0 & \binom{3}{0}+\binom{4}{1} & \binom{3}{2}+\binom{4}{3} & 0 & 0 & 0 & 0 \\
p^3 & 0 & \binom{5}{0} & \binom{4}{1}+\binom{5}{2} & \binom{4}{3}+\binom{5}{4} & 0 & 0 & 0 \\
p^4 & 0 & 0 & \binom{5}{0}+\binom{6}{1} & \binom{5}{2}+\binom{6}{3} & \binom{5}{4}+\binom{6}{5} & 0 & 0 \\
p^5 & 0 & 0 & \binom{7}{0} & \binom{6}{1}+\binom{7}{2} & \binom{6}{3}+\binom{7}{4} & \binom{6}{5}+\binom{7}{6} & 0 \\
p^6 & 0 & 0 & 0 & \binom{7}{0}+\binom{8}{1} & \binom{7}{2}+\binom{8}{3} & \binom{7}{4}+\binom{8}{5} & \binom{7}{6}+\binom{8}{7} \\
p^7 & 0 & 0 & 0 & \binom{9}{0} & \binom{8}{1}+\binom{9}{2} & \binom{8}{3}+\binom{9}{4} & \binom{8}{5}+\binom{9}{6}
\end{vmatrix}$$

kxk determinant

$p = n(n+1)$

(52)

6.1.2 Sum of odd powers

$$\frac{f(n,2k+1)}{f(n,3)} = \frac{\sum_{i=1}^{n} i^{2k+1}}{\sum_{i=1}^{n} i^{3}} =$$

$$\frac{2(-1)^{k+1}}{(k+1)!}\begin{vmatrix}
p^0 & \binom{2}{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p^1 & \binom{3}{0} & \binom{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p^2 & 0 & \binom{4}{1} & \binom{4}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
p^3 & 0 & \binom{5}{0} & \binom{5}{2} & \binom{5}{4} & 0 & 0 & 0 & 0 & 0 \\
p^4 & 0 & 0 & \binom{6}{1} & \binom{6}{3} & \binom{6}{5} & 0 & 0 & 0 & 0 \\
p^5 & 0 & 0 & \binom{7}{0} & \binom{7}{2} & \binom{7}{4} & \binom{7}{6} & 0 & 0 & 0 \\
p^6 & 0 & 0 & 0 & \binom{8}{1} & \binom{8}{3} & \binom{8}{5} & \binom{8}{7} & 0 & 0 \\
p^7 & 0 & 0 & 0 & \binom{9}{0} & \binom{9}{2} & \binom{9}{4} & \binom{9}{6} & \binom{9}{8} & 0 \\
p^8 & 0 & 0 & 0 & 0 & \binom{10}{1} & \binom{10}{3} & \binom{10}{5} & \binom{10}{7} & \binom{10}{9} \\
p^9 & 0 & 0 & 0 & 0 & \binom{11}{0} & \binom{11}{2} & \binom{11}{4} & \binom{11}{6} & \binom{11}{8}
\end{vmatrix}$$

kxk determinant

$p = n(n+1)$

(53)

<u>6.2 ±oscillating sum</u>

$$os(n,m) = \sum_{i=1}^{n} (-1)^{n-i} i^m \ ; \quad os(n,0) = \begin{cases} 0 \\ 1 \end{cases} \text{for n is} \begin{cases} \text{even} \\ \text{odd} \end{cases} ;$$

$$os(n,1) = int(\frac{n+1}{2}) \ ; \qquad (54)$$

$$os(n,2) = \frac{1}{2} n(n+1) = s(n,1)$$

6.2.1 ±oscillating sum of even powers

$$\frac{os(n,2k)}{os(n,2)}=\frac{\sum_{i=1}^{n}(-1)^{n-i}i^{2k}}{\sum_{i=1}^{n}(-1)^{n-i}i^{2}}$$

$$= (-1)^{k+1}\begin{vmatrix}
p^0 & \binom{1}{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p^1 & \binom{2}{0} & \binom{2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p^2 & 0 & \binom{3}{1} & \binom{3}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
p^3 & 0 & \binom{4}{0} & \binom{4}{2} & \binom{4}{4} & 0 & 0 & 0 & 0 & 0 \\
p^4 & 0 & 0 & \binom{5}{1} & \binom{5}{3} & \binom{5}{5} & 0 & 0 & 0 & 0 \\
p^5 & 0 & 0 & \binom{6}{0} & \binom{6}{2} & \binom{6}{4} & \binom{6}{6} & 0 & 0 & 0 \\
p^6 & 0 & 0 & 0 & \binom{7}{1} & \binom{7}{3} & \binom{7}{5} & \binom{7}{7} & 0 & 0 \\
p^7 & 0 & 0 & 0 & \binom{8}{0} & \binom{8}{2} & \binom{8}{4} & \binom{8}{6} & \binom{8}{8} & 0 \\
p^8 & 0 & 0 & 0 & 0 & \binom{9}{1} & \binom{9}{3} & \binom{9}{5} & \binom{9}{7} & \binom{9}{9} \\
p^9 & 0 & 0 & 0 & 0 & \binom{10}{0} & \binom{10}{2} & \binom{10}{4} & \binom{10}{6} & \binom{10}{8}
\end{vmatrix}$$

kxk determinant

$p = n(n+1)$

(55)

6.2.2 ±oscillating sum of odd powers

Open problem

6.3 Other oscillation patterns sum

7. Extended quotient rule in an algorithmic-oriented representation (array + navigation rule + local operator)

Cartesian mxm-array (2D pattern processing algorithms), here m=6

Initial setting:

6	$U^{(4)}$	$V^{(4)}$	$V^{(3)}$	V''	V'	V
5	$U^{(3)}$	$V^{(3)}$	V''	V'	V	0
4	U''	V''	V'	V	0	0
3	U'	V'	V	0	0	0
2	U	V	0	0	0	0
1	0	0	0	0	0	0
	1	2	3	4	5	6

Algorithm for the simultaneous sequential calculation of $F^{(0)}$, $F^{(1)}$, $F^{(2)}$, $F^{(3)}$, $F^{(4)}$, …

b_0:=U

$$F^{(0)} := \frac{b_0}{V}$$

j=2(1)m-1 navigation

k=j+1(1)m rule

c[1,k]:=c[j,j]*c[1,k]-c[1,j]*c[j,k] local operator

new old content of cell[1,k]

b_{j-1}:=c[1,j+1]

$$F^{(j-1)} := \frac{b_{j-1}}{V^{j}}$$

Starting from j=2 by step 1
the old column[1] above j is overwritten with
(content of diagonal cell[j,j] times old column[1] above j minus content of cell[1,j] times column[j] above j).

A very important algorithm for many applications! What happens when we repeat this procedure?

Final setting of the array

6	$b_4 = V(V(V(VU^{(4)} - b_0V^{(4)}) - b_1V^{(3)}) - b_2V'') - b_3V'$ $V(V(VU^{(4)} - b_0V^{(4)}) - b_1V^{(3)}) - b_2V''$ $V(VU^{(4)} - b_0V^{(4)}) - b_1V^{(3)}$ $VU^{(4)} - b_0V^{(4)}$	$V^{(4)}$	$V^{(3)}$	V''	V'	V
5	$b_3 = V(V(VU^{(3)} - b_0V^{(3)}) - b_1V'') - b_2V'$ $V(VU^{(3)} - b_0V^{(3)}) - b_1V''$ $VU^{(3)} - b_0V^{(3)}$	$V^{(3)}$	V''	V'	V	0
4	$b_2 = V(VU'' - b_0V'') - b_1V'$ $VU'' - b_0V''$	V''	V'	V	0	0
3	$b_1 = VU' - b_0V'$	V'	V	0	0	0
2	$b_0 = U$	V	0	0	0	0
1	0	0	0	0	0	0
	1	2	3	4	5	6

$$b_4 = ((((U^{(4)})V - b_0V^{(4)})V - b_1V^{(3)})V - b_2V'')V - b_3V'$$

$$= U^{(4)}VVVV - U^{(3)}VVVV' - U''VVVV'' + U''VVV'V' - U'VVVV^{(3)}$$

$$+U'VVV'V'' + U'VVV'V'' - U'VV'V'V' - UVVVV^{(4)} + UVVV'V^{(3)}$$

$$+UVVV'V^{(3)} + UVVV''V'' - UVV'V'V'' - UVV'V'V'' - UVV'V'V''$$

$$+UV'V'V'V'$$

$$b_3 = (((U^{(3)})V - b_0V^{(3)})V - b_1V'')V - b_2V'$$

$$= U^{(3)}VVV - U''VVV' - U'VVV'' + U'VV'V' - UVVV^{(3)} + UVV'V''$$

$$+UVV'V''-UV'V'V'$$

$$b_2 = ((U'')V - b_0V'')V - b_1V'$$

$$= U''VV - U'VV' - UVV'' + UV'V'$$

$$b_1 = (U')V - b_0V'$$

$$= U'V - UV'$$

$$b_0 = U$$

8. Applications

A **prospective application area** in physics and engineering is the systematic design of static and dynamic, stationary and movable discrete structures (esp. metastructures with negative refractive index; 2D → 3D; macro, micro, nano range; +global scaling?

http://www.global-scaling-institute.de

Global Scaling – Basis eines neuen wissenschaftlichen Weltbildes. München 2009, ISBN 978-3-940965-21-9

which are able to guide a continuous wave or dynamic field around an object without considerable energetic interaction, impact absorption, destruction, interference, and observation.

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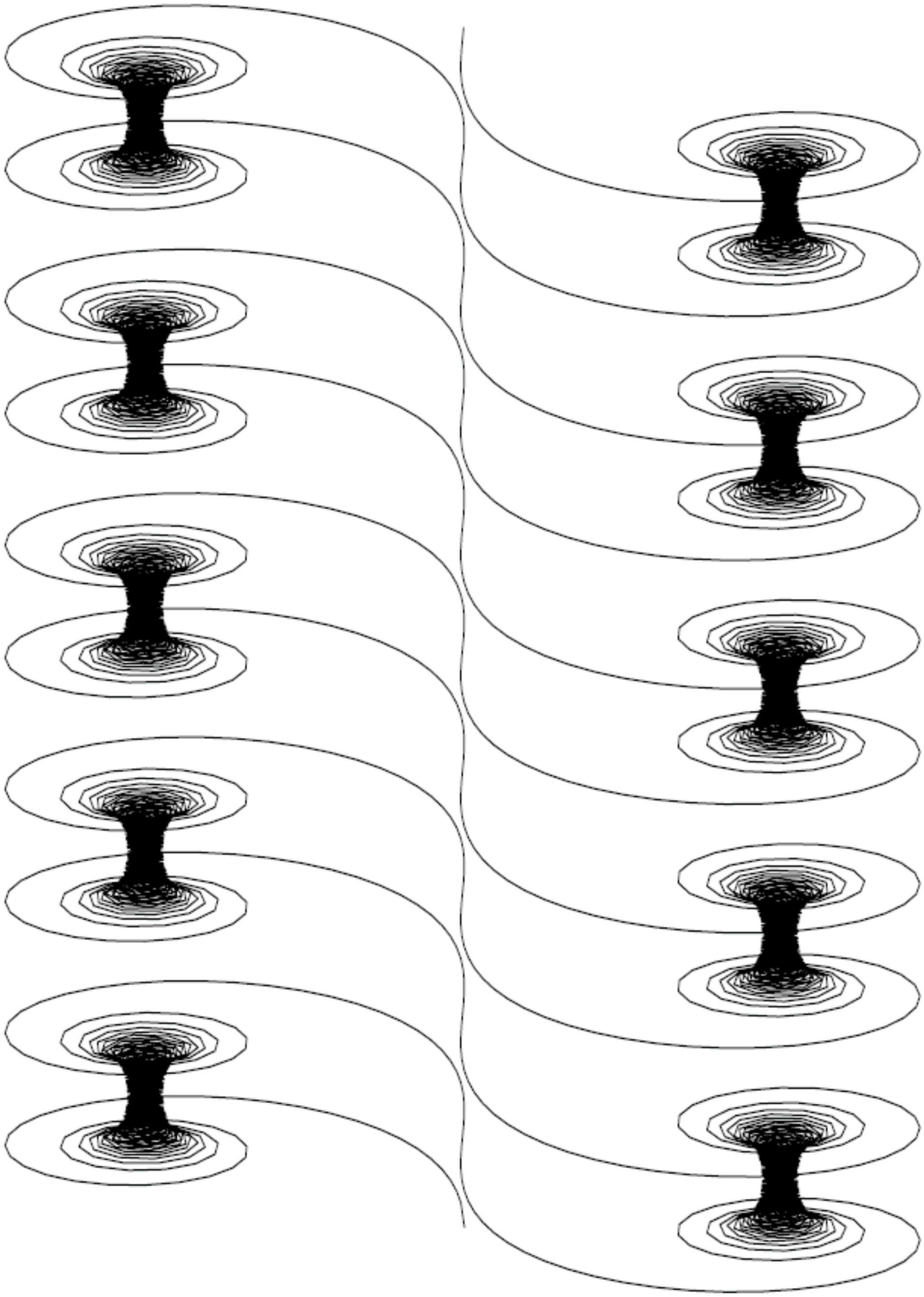
References available



Coupled rotational field